Multiple Integrals: Change of Coordinates

Calculus III

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TTU

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Change of Coordinates in Two Variables

Proposition

Let $D \subset \mathbb{R}^2$ be a closed & bounded region in the xy-plane.
Let $D^* \subset \mathbb{R}^2$ be a closed & bounded region in the uv-plane.
Let $f \in C(D)$.
Let transformation T map region D to region D^* s.t.

$$T : \begin{cases} x = T_1(u,v) \\ y = T_2(u,v) \end{cases} \quad \text{where } T_1, T_2 \in C^{(1,1)}(D^*)$$

Then:

$$\iint_D f(x,y)\, dA = \iint_{D^*} f\left[T_1(u,v), T_2(u,v)\right] \left| \frac{\partial(x,y)}{\partial(u,v)} \right| dA$$

where $\dfrac{\partial(x,y)}{\partial(u,v)} := \det \begin{bmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{bmatrix}$ *is called the* ***Jacobian***.

PROOF: Take Linear Algebra & Advanced Calculus.

The Jacobian (Rectangular $\to$ Polar Coordinates)

Let transformation T map from Rectangular $\to$ Polar $\quad T : \begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases}$

Then:

$$\begin{aligned}
\frac{\partial(x,y)}{\partial(r,\theta)} &= \det \begin{bmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{bmatrix} \\
&= \det \begin{bmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{bmatrix} \\
&= r\cos^2\theta + r\sin^2\theta \\
&= r\left(\cos^2\theta + \sin^2\theta\right) \\
&= r
\end{aligned}$$

$$\begin{aligned}
\therefore \iint_D f(x,y)\, dA &= \iint_{D^*} f(r\cos\theta, r\sin\theta) \left|\frac{\partial(x,y)}{\partial(r,\theta)}\right| \, dr\, d\theta \\
&= \iint_{D^*} f(r\cos\theta, r\sin\theta)\, r\, dr\, d\theta
\end{aligned}$$

Change of Coordinates in Two Variables

Sometimes it's easier to write a transformation in terms of x, y instead of u, v like so:

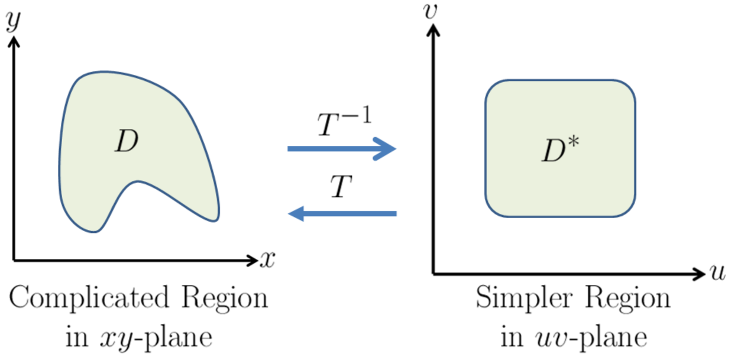
$$T^{-1} : \begin{cases} u = T_1^{-1}(x, y) \\ v = T_2^{-1}(x, y) \end{cases} \quad \text{where } T_1^{-1}, T_2^{-1} \in C^{(1,1)}(D)$$

In other words, sometimes it's easier to work with an **inverse transformation** (instead of a transformation.)

Then, to compute the Jacobian $\dfrac{\partial(x, y)}{\partial(u, v)}$:

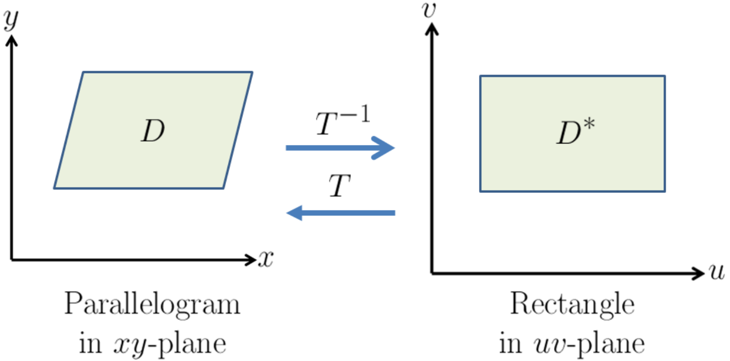
1. Compute Jacobian $\dfrac{\partial(u, v)}{\partial(x, y)} := \det \begin{bmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{bmatrix}$
2. Solve equation $\dfrac{\partial(x, y)}{\partial(u, v)} \cdot \dfrac{\partial(u, v)}{\partial(x, y)} = 1$ for $\dfrac{\partial(x, y)}{\partial(u, v)}$

Transformation T may Simplify Region of Integration



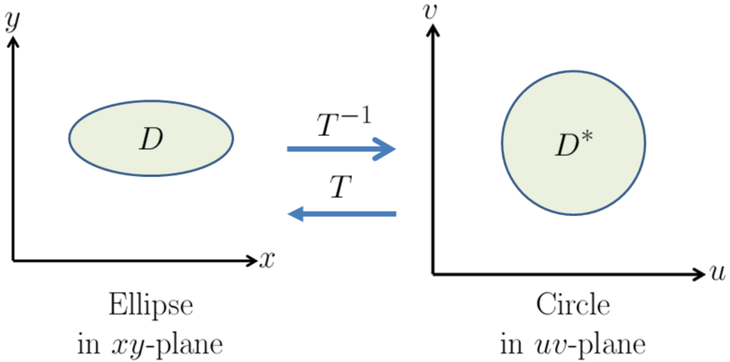
Notice that transformation T maps from the uv-plane to the xy-plane
(not the other way around as one would have expected.)

Transformation T may Simplify Region of Integration



Notice that transformation T maps from the uv-plane to the xy-plane (not the other way around as one would have expected.)

Transformation T may Simplify Region of Integration



Notice that transformation T maps from the uv-plane to the xy-plane (not the other way around as one would have expected.)

Inverse Transformation T^{-1} may Simplify the Integrand

Let transformation $T^{-1}: \begin{cases} u = 2x - 3y \\ v = 2x + 3y \end{cases}$. Then:

Jacobian $\dfrac{\partial(u,v)}{\partial(x,y)} = \det \begin{bmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{bmatrix} = \det \begin{bmatrix} 2 & -3 \\ 2 & 3 \end{bmatrix} = (2)(3) - (-3)(2) = 12$

Now, $\dfrac{\partial(u,v)}{\partial(x,y)} \cdot \dfrac{\partial(x,y)}{\partial(u,v)} = 1 \implies (12)\dfrac{\partial(x,y)}{\partial(u,v)} = 1 \implies \dfrac{\partial(x,y)}{\partial(u,v)} = \dfrac{1}{12}$

$$\therefore \iint_D \sqrt{\frac{2x-3y}{2x+3y}}\, dA \overset{CV}{=} \iint_{D^*} \sqrt{\frac{u}{v}} \left| \frac{\partial(x,y)}{\partial(u,v)} \right| dA = \frac{1}{12} \iint_{D^*} \sqrt{\frac{u}{v}}\, dA$$

Change of Coordinates in Three Variables

Proposition

Let $E \subset \mathbb{R}^3$ be a closed & bounded solid in xyz-space.
Let $E^* \subset \mathbb{R}^3$ be a closed & bounded solid in uvw-space.
Let $f \in C(E)$.
Let transformation T map solid E to solid E^* s.t.

$$T : \begin{cases} x = T_1(u,v,w) \\ y = T_2(u,v,w) \\ z = T_3(u,v,w) \end{cases} \quad \text{where } T_1, T_2, T_3 \in C^{(1,1,1)}(E^*)$$

Then:

$$\iiint_E f(x,y,z)\, dV = \iiint_{E^*} f\left[T_1(u,v,w), T_2(u,v,w), T_3(u,v,w)\right] \left| \frac{\partial(x,y,z)}{\partial(u,v,w)} \right| \, dV$$

where $\dfrac{\partial(x,y,z)}{\partial(u,v,w)} := \det \begin{bmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{bmatrix}$ *is called the **Jacobian**.*

PROOF: Take Linear Algebra & Advanced Calculus.

The Jacobian (Rectangular $\to$ Cylindrical Coordinates)

Let T map from Rectangular $\to$ Cylindrical $\quad T : \begin{cases} x = r\cos\theta \\ y = r\sin\theta \\ z = z \end{cases}$

Then:
$$\frac{\partial(x,y,z)}{\partial(r,\theta,z)} = \det\begin{bmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial z} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial z} \\ \frac{\partial z}{\partial r} & \frac{\partial z}{\partial \theta} & \frac{\partial z}{\partial z} \end{bmatrix}$$
$$= \det\begin{bmatrix} \cos\theta & -r\sin\theta & 0 \\ \sin\theta & r\cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$
$$= \cos\theta \begin{vmatrix} r\cos\theta & 0 \\ 0 & 1 \end{vmatrix} - (-r\sin\theta)\begin{vmatrix} \sin\theta & 0 \\ 0 & 1 \end{vmatrix} + 0$$
$$= r\cos^2\theta + r\sin^2\theta$$
$$= r\Big(\cos^2\theta + \sin^2\theta\Big)$$
$$= r$$

The Jacobian (Rectangular $\to$ Spherical Coordinates)

Let T map from Rectangular $\to$ Spherical $\quad T : \begin{cases} x = \rho \sin\phi \cos\theta \\ y = \rho \sin\phi \sin\theta \\ z = \rho \cos\phi \end{cases}$

$$\begin{aligned}
\frac{\partial(x,y,z)}{\partial(\rho,\theta,\phi)} &= \det \begin{bmatrix} \frac{\partial x}{\partial \rho} & \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial \phi} \\ \frac{\partial y}{\partial \rho} & \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial \phi} \\ \frac{\partial z}{\partial \rho} & \frac{\partial z}{\partial \theta} & \frac{\partial z}{\partial \phi} \end{bmatrix} \\
&= \det \begin{bmatrix} \sin\phi\cos\theta & -\rho\sin\phi\sin\theta & \rho\cos\phi\cos\theta \\ \sin\phi\sin\theta & \rho\sin\phi\cos\theta & \rho\cos\phi\sin\theta \\ \cos\phi & 0 & -\rho\sin\phi \end{bmatrix} \\
&= \begin{matrix} -\rho^2\sin^3\phi\cos^2\theta - \rho^2\sin^3\phi\sin^2\theta \\ -\rho^2\cos^2\phi\sin\phi\sin^2\theta - \rho^2\cos^2\phi\sin\phi\cos^2\theta \end{matrix} \\
&= -\rho^2\sin^3\phi\left[\cos^2\theta + \sin^2\theta\right] - \rho^2\cos^2\phi\sin\phi\left[\sin^2\theta + \cos^2\theta\right] \\
&= -\rho^2\sin^3\phi - \rho^2\cos^2\phi\sin\phi \\
&= -\rho^2\sin\phi\left[\sin^2\phi + \cos^2\phi\right] \\
&= -\rho^2\sin\phi
\end{aligned}$$

Change of Coordinates in Three Variables

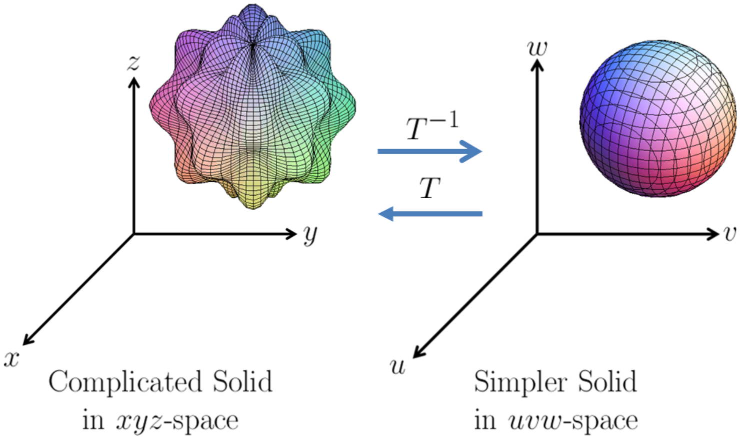
Sometimes it's easier to write a transformation in terms of x, y, z instead of u, v, w like so:

$$T^{-1} : \begin{cases} u = T_1^{-1}(x, y, z) \\ v = T_2^{-1}(x, y, z) \\ w = T_3^{-1}(x, y, z) \end{cases} \quad \text{where } T_1^{-1}, T_2^{-1}, T_3^{-1} \in C^{(1,1,1)}(E)$$

Then, to compute the Jacobian $\dfrac{\partial(x, y, z)}{\partial(u, v, w)}$:

1. Compute Jacobian $\dfrac{\partial(u, v, w)}{\partial(x, y, z)} := \det \begin{bmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z} \end{bmatrix}$
2. Solve equation $\dfrac{\partial(x, y, z)}{\partial(u, v, w)} \cdot \dfrac{\partial(u, v, w)}{\partial(x, y, z)} = 1$ for $\dfrac{\partial(x, y, z)}{\partial(u, v, w)}$

Transformation T may Simplify Solid of Integration



Notice that transformation T maps from uvw-space to xyz-space
(not the other way around as one would have expected.)

Linear Transformations in $\mathbb{R}^2$

Definition

(Linear Transformation in $\mathbb{R}^2$)

Transformation T is **linear** if $T : \begin{cases} x = au + bv \\ y = cu + dv \end{cases}$, where $a, b, c, d \in \mathbb{R}$

Inverse transformation T^{-1} is **linear** if $T^{-1} : \begin{cases} u = ax + by \\ v = cx + dy \end{cases}$, with $a, b, c, d \in \mathbb{R}$

Theorem

Transformation T is linear $\iff$ inverse transformation T^{-1} is linear.

Typical examples of linear transformations:

- Dilations
- Rotations
- Relections
- Shears

NOTE: This is an overview of linear transformations – take **Linear Algebra**.

Linear Transformations in $\mathbb{R}^3$

Definition

(Linear Transformation in $\mathbb{R}^3$)

Transformation T is **linear** if $T: \begin{cases} x = a_{11}u + a_{12}v + a_{13}w \\ y = a_{21}u + a_{22}v + a_{23}w \\ z = a_{31}u + a_{32}v + a_{33}w \end{cases}$, with $a_{11}, ..., a_{33} \in \mathbb{R}$

Inverse trans. T^{-1} is **linear** if $T^{-1}: \begin{cases} u = a_{11}x + a_{12}y + a_{13}z \\ v = a_{21}x + a_{22}y + a_{23}z \\ w = a_{31}x + a_{32}y + a_{33}z \end{cases}$, $a_{11}, ..., a_{33} \in \mathbb{R}$

Theorem

Transformation T is linear $\iff$ inverse transformation T^{-1} is linear.

Typical examples of linear transformations:

- Dilations
- Rotations
- Relections
- Shears

NOTE: This is an overview of linear transformations – take **Linear Algebra**.

Double Integrals (Coordinate Change Procedure)

SETUP: Let transformation T map region D^* to region D s.t.

$$T : \begin{cases} x = T_1(u,v) \\ y = T_2(u,v) \end{cases} \quad \text{where } T_1, T_2 \in C^{(1,1)}(D^*)$$

1. Sketch region D in xy-plane & label BC's
2. Apply transformation to each BC of region D to obtain a BC of region D^*
3. Sketch region D^* in uv-plane & label BC's & BP's
4. Compute the Jacobian $\dfrac{\partial(x,y)}{\partial(u,v)}$ in terms of u & v.
5. Compute the **absolute value** of Jacobian $\left|\dfrac{\partial(x,y)}{\partial(u,v)}\right|$
6. $\displaystyle\iint_D f(x,y)\,dA = \iint_{D^*} f\left[T_1(u,v), T_2(u,v)\right] \left|\frac{\partial(x,y)}{\partial(u,v)}\right| \,dA$

Double Integrals (Coordinate Change Procedure)

SETUP: Let inverse transformation T^{-1} map region D to region D^* s.t.

$$T^{-1}: \begin{cases} u = T_1^{-1}(x,y) \\ v = T_2^{-1}(x,y) \end{cases} \quad \text{where } T_1^{-1}, T_2^{-1} \in C^{(1,1)}(D)$$

1. Sketch region D in xy-plane & label BC's
2. Apply transformation to each BC of region D to obtain a BC of region D^*
   - Trivial if BC's of D are of the form: $T_1^{-1}(x,y) = k_1, T_1^{-1}(x,y) = k_2, T_2^{-1}(x,y) = k_3, \ldots,$ where $k_1, k_2, k_3, \cdots \in \mathbb{R}$
   - If T^{-1} is **linear**, solve linear system to obtain transformation T in terms of u, v.
3. Sketch region D^* in uv-plane & label BC's & BP's
4. Compute Jacobian $\frac{\partial(x,y)}{\partial(u,v)}$ in terms of u & v by solving $\frac{\partial(x,y)}{\partial(u,v)} \cdot \frac{\partial(u,v)}{\partial(x,y)} = 1$
5. Compute the **absolute value** of Jacobian $\left|\frac{\partial(x,y)}{\partial(u,v)}\right|$
6. $\iint_D f(x,y)\, dA = \iint_{D^*} f\left[T_1(u,v), T_2(u,v)\right] \left|\frac{\partial(x,y)}{\partial(u,v)}\right| \, dA$

Fin.