Vector Fields: Intro, Div, Curl

Calculus III

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07 November 2014

"The Function Landscape"

FUNCTION TYPE	PROTOTYPE	MAPPING
(Scalar) Function	$y = f(x)$	f maps scalar $\to$ scalar
2D Vector Function 3D Vector Function	$\mathbf{F}(t) = \langle f_1(t), f_2(t) \rangle$ $\mathbf{F}(t) = \langle f_1(t), f_2(t), f_3(t) \rangle$	$\mathbf{F}$ maps scalar $\to$ 2D vector $\mathbf{F}$ maps scalar $\to$ 3D vector
Function of 2 Variables Function of 3 Variables	$z = f(x, y)$ $w = f(x, y, z)$	f maps 2D point $\to$ scalar f maps 3D point $\to$ scalar

2D Vector Field	$\mathbf{F}(x, y) = \langle M(x, y), N(x, y) \rangle$	$\mathbf{F}$ maps 2D point to 2D vector
3D Vector Field	$\mathbf{F}(x, y, z) = \langle M(x, y, z), N(x, y, z), P(x, y, z) \rangle$	$\mathbf{F}$ maps 3D point to 3D vector

REMARK: Going forward, multivariable functions $f(x, y)$, $g(x, y, z)$ may be referred to as **scalar fields**.

"The Function Landscape"

FUNCTION TYPE	PROTOTYPE	MAPPING
(Scalar) Function	$y = f(x)$	$f : \mathbb{R} \to \mathbb{R}$
2D Vector Function	$\mathbf{F}(t) = \langle f_1(t), f_2(t) \rangle$	$\mathbf{F} : \mathbb{R} \to \mathbb{R}^2$
3D Vector Function	$\mathbf{F}(t) = \langle f_1(t), f_2(t), f_3(t) \rangle$	$\mathbf{F} : \mathbb{R} \to \mathbb{R}^3$
Function of 2 Variables	$z = f(x, y)$	$f : \mathbb{R}^2 \to \mathbb{R}$
Function of 3 Variables	$w = f(x, y, z)$	$f : \mathbb{R}^3 \to \mathbb{R}$

2D Vector Field	$\mathbf{F}(x, y) = \langle M(x, y), N(x, y) \rangle$	$\mathbf{F} : \mathbb{R}^2 \to \mathbb{R}^2$
3D Vector Field	$\mathbf{F}(x, y, z) = \langle M(x, y, z), N(x, y, z), P(x, y, z) \rangle$	$\mathbf{F} : \mathbb{R}^3 \to \mathbb{R}^3$

$\mathbb{R} :=$ The set of all scalars on the real line

$\mathbb{R}^2 :=$ The set of all ordered pairs (x, y) on the xy-plane (say "R Two")

$\mathbb{R}^2 :=$ The set of all 2D vectors $\langle v_1, v_2 \rangle$ on the xy-plane

$\mathbb{R}^3 :=$ The set of all ordered triples (x, y, z) in xyz-space (say "R Three")

$\mathbb{R}^3 :=$ The set of all 3D vectors $\langle v_1, v_2, v_3 \rangle$ in xyz-space

Vector Fields in $\mathbb{R}^2$

Definition

A **2D vector field** is a function $\mathbf{F}(x,y)$ that assigns a vector to each point in its domain and has the form

$$\mathbf{F}(x,y) = \langle M(x,y), N(x,y) \rangle = M(x,y)\widehat{\mathbf{i}} + N(x,y)\widehat{\mathbf{j}}$$

where the components of $\mathbf{F}$ are (scalar) functions of two variables.

Proposition

Let $D \subset \mathbb{R}^2$ be a region on the xy-plane.
Let $\mathbf{F}(x,y) = \langle M(x,y), N(x,y) \rangle$ be a 2D vector field. Then:

- *Dom*$(\mathbf{F}) :=$ *Dom*$(M) \cap$ *Dom*(N)
- $\mathbf{F}(x,y)$ *is* ***continuous*** *on region* $D \iff$ *its components are continous:*

$$\mathbf{F}(x,y) \in C(D) \iff M(x,y), N(x,y) \in C(D)$$

- $\mathbf{F}(x,y) \in C^{(1,1)}(D) \iff M(x,y), N(x,y) \in C^{(1,1)}(D)$
- $\mathbf{F}(x,y) \in C^{(2,2)}(D) \iff M(x,y), N(x,y) \in C^{(2,2)}(D)$

Vector Fields in $\mathbb{R}^3$

Definition

A **3D vector field** is a function $\mathbf{F}(x, y, z)$ that assigns a vector to each point in its domain and has the form

$$\mathbf{F}(x,y,z) = \langle M(x,y,z), N(x,y,z), P(x,y,z)\rangle = M(x,y,z)\widehat{\mathbf{i}} + N(x,y,z)\widehat{\mathbf{j}} + P(x,y,z)\widehat{\mathbf{k}}$$

where the components of $\mathbf{F}$ are (scalar) functions of three variables.

Proposition

Let $E \subset \mathbb{R}^3$ be a solid in xyz-space.
Let $\mathbf{F}(x,y,z) = \langle M(x,y,z), N(x,y,z), P(x,y,z)\rangle$ be a 3D vector field. Then:

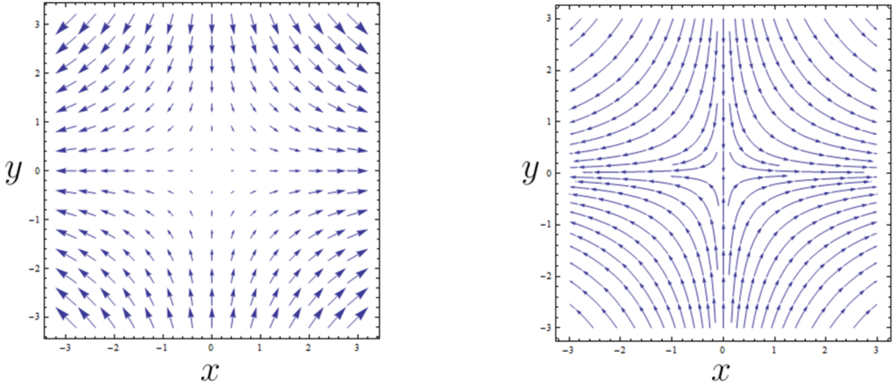
- *Dom*$(\mathbf{F}) :=$ *Dom*$(M) \cap$ *Dom*$(N) \cap$ *Dom*(P)
- $\mathbf{F}(x,y,z)$ *is **continuous** on solid E $\iff$ its components are continous:*

$$\mathbf{F}(x,y,z) \in C(E) \iff M(x,y,z), N(x,y,z), P(x,y,z) \in C(E)$$

- $\mathbf{F}(x,y,z) \in C^{(1,1,1)}(E) \iff M(x,y,z), N(x,y,z), P(x,y,z) \in C^{(1,1,1)}(E)$
- $\mathbf{F}(x,y,z) \in C^{(2,2,2)}(E) \iff M(x,y,z), N(x,y,z), P(x,y,z) \in C^{(2,2,2)}(E)$

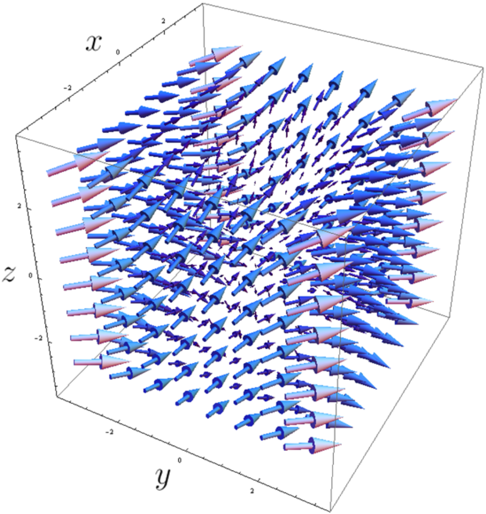
Vector Fields in $\mathbb{R}^2$ (Plot)

$$\vec{\mathbf{F}}(x, y) = \langle 10x, -10y \rangle = 10x\hat{\mathbf{i}} - 10y\hat{\mathbf{j}}$$

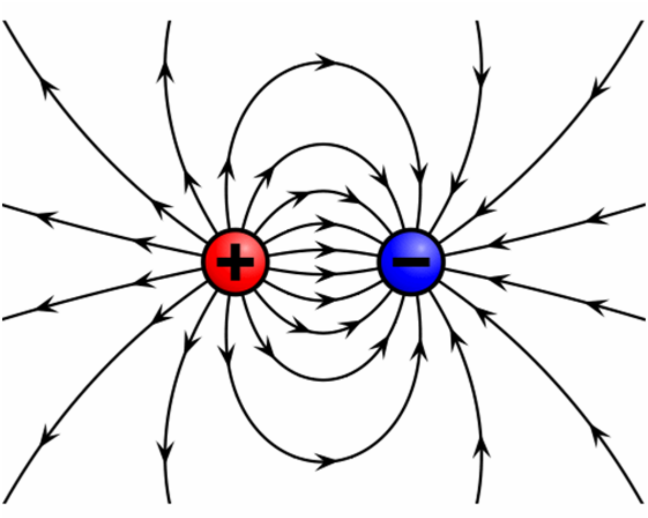


Vector Plot   Stream Plot

$$\vec{\mathbf{F}}(x, y, z) = \langle x^2, y^2, z \rangle = \left(x^2\right)\widehat{\mathbf{i}} + \left(y^2\right)\widehat{\mathbf{j}} + z\widehat{\mathbf{k}}$$

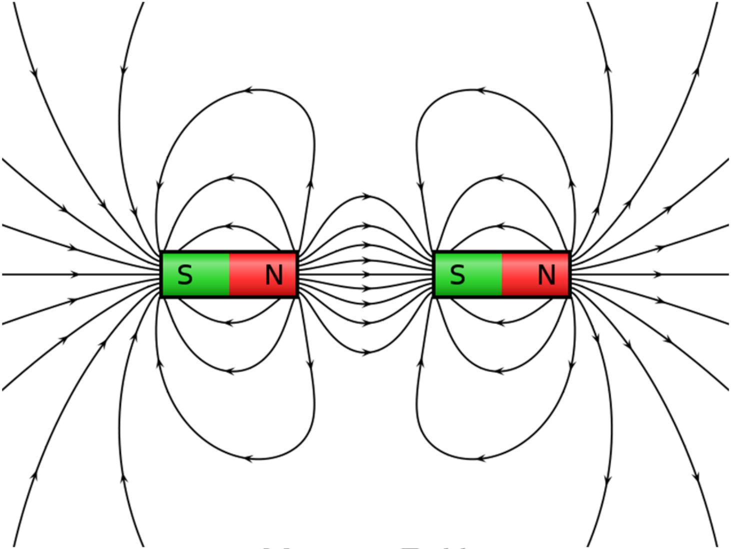


Examples of Vector Fields



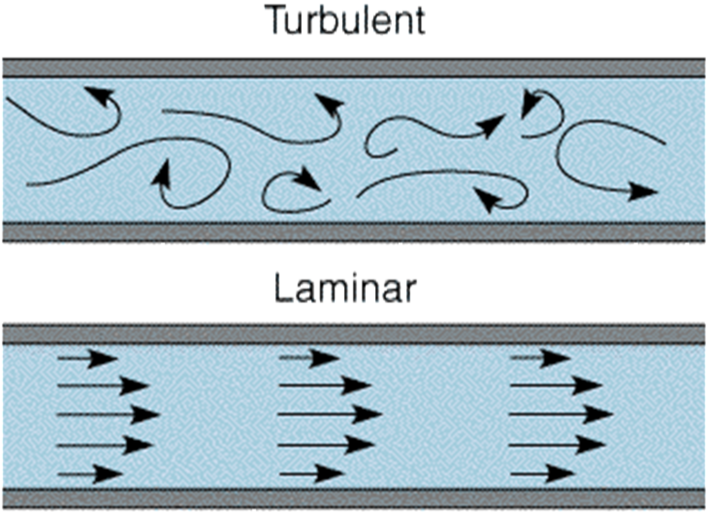
Electric Field

Examples of Vector Fields



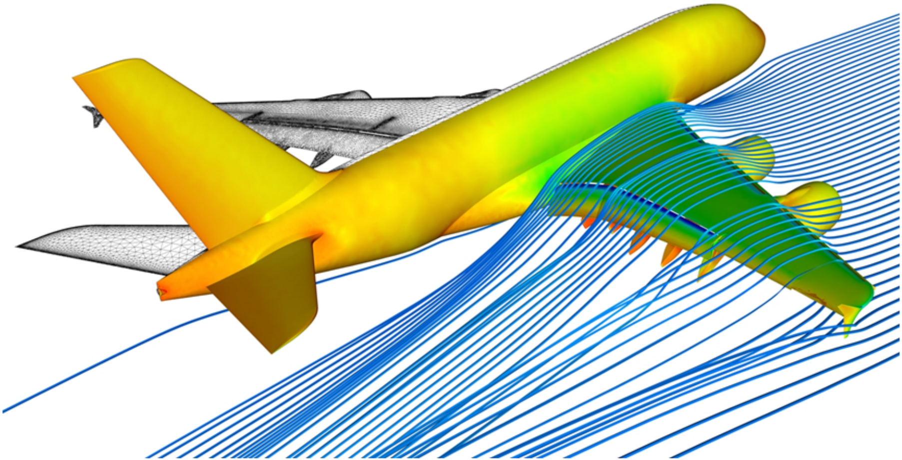
Magnetic Field

Examples of Vector Fields



Fluid Flow

Examples of Vector Fields



Fluid Flow

WEX 13-1-1: Let $\mathbf{F}(x,y,z) = \langle xy, yz, xz \rangle$ and $\mathbf{G}(x,y,z) = \langle x, 2y, 3z \rangle$.
Compute: (a) $\mathbf{F}+\mathbf{G}$ (b) $\mathbf{F}-\mathbf{G}$ (c) $\mathbf{F}\cdot\mathbf{G}$ (d) $\mathbf{F}\times\mathbf{G}$

(a) $\mathbf{F}+\mathbf{G} = \langle xy, yz, xz \rangle + \langle x, 2y, 3z \rangle = \boxed{\langle xy+x, yz+2y, xz+3z \rangle}$

(b) $\mathbf{F}-\mathbf{G} = \langle xy, yz, xz \rangle - \langle x, 2y, 3z \rangle = \boxed{\langle xy-x, yz-2y, xz-3z \rangle}$

(c)
$$\mathbf{F}\cdot\mathbf{G} = \langle xy, yz, xz \rangle \cdot \langle x, 2y, 3z \rangle = (xy)(x) + (yz)(2y) + (xz)(3z) = \boxed{x^2y + 2y^2z + 3xz^2}$$

(d)
$$\begin{aligned}
\mathbf{F}\times\mathbf{G} &= \langle xy, yz, xz \rangle \times \langle x, 2y, 3z \rangle \\
&= \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ xy & yz & xz \\ x & 2y & 3z \end{vmatrix} \\
&= \begin{vmatrix} yz & xz \\ 2y & 3z \end{vmatrix} \hat{\mathbf{i}} - \begin{vmatrix} xy & xz \\ x & 3z \end{vmatrix} \hat{\mathbf{j}} + \begin{vmatrix} xy & yz \\ x & 2y \end{vmatrix} \hat{\mathbf{k}} \\
&= [(yz)(3z) - (xz)(2y)]\,\hat{\mathbf{i}} - [(xy)(3z) - (xz)(x)]\,\hat{\mathbf{j}} + [(xy)(2y) - (yz)(x)]\,\hat{\mathbf{k}} \\
&= \boxed{\langle 3yz^2 - 2xyz, x^2 - 3xyz, 2xy^2 - xyz \rangle}
\end{aligned}$$

The Del Operator ∇

Definition

The **del operator** in $\mathbb{R}^2$ is defined by

$$\nabla := \left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y} \right\rangle$$

Definition

The **del operator** in $\mathbb{R}^3$ is defined by

$$\nabla := \left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right\rangle$$

i.e., ∇ is the vector with the partial derivative operators as components.

REMARK: The del operator is useful in simplifying notation & remembering certain vector field operations.

Divergence of a Vector Field

Definition

Let 2D vector field $\mathbf{F}(x, y) \in C^{(1,1)}$ s.t. $\mathbf{F}(x, y) = \langle M(x, y), N(x, y)\rangle$.
Then the **divergence** of $\mathbf{F}(x, y)$ is

$$\text{div } \mathbf{F} \equiv \nabla \cdot \mathbf{F} = \left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y} \right\rangle \cdot \langle M, N\rangle := \frac{\partial M}{\partial x} + \frac{\partial N}{\partial y} = M_x + N_y$$

Definition

Let 3D vector field $\mathbf{F}(x, y, z) \in C^{(1,1,1)}$ s.t.
$\mathbf{F}(x, y, z) = \langle M(x, y, z), N(x, y, z), P(x, y, z)\rangle$.

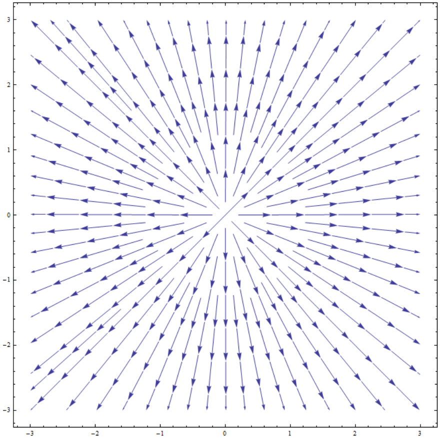
Then the **divergence** of $\mathbf{F}(x, y, z)$ is

$$\text{div } \mathbf{F} \equiv \nabla \cdot \mathbf{F} = \left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right\rangle \cdot \langle M, N, P\rangle := \frac{\partial M}{\partial x} + \frac{\partial N}{\partial y} + \frac{\partial P}{\partial z} = M_x + N_y + P_z$$

<u>REMARK:</u> The divergence of a vector field is a scalar field.

Positive Divergence (Geometric Interpretation)

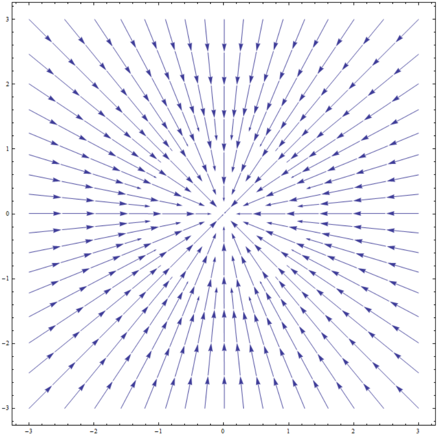
$$\vec{\mathbf{F}}(x,y) = \langle 10x, 10y \rangle$$



$$\nabla \cdot \vec{\mathbf{F}} = \frac{\partial}{\partial x}\left[10x\right] + \frac{\partial}{\partial y}\left[10y\right] = 10 + 10 = 20 > 0$$

Negative Divergence (Geometric Interpretation)

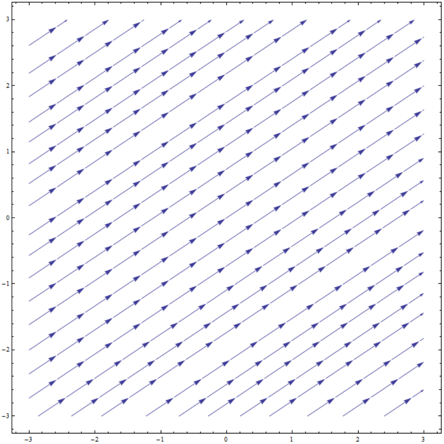
$$\vec{\mathbf{F}}(x, y) = \langle -10x, -10y \rangle$$



$$\nabla \cdot \vec{\mathbf{F}} = \frac{\partial}{\partial x}\Big[-10x\Big] + \frac{\partial}{\partial y}\Big[-10y\Big] = -10 - 10 = -20 < 0$$

Zero Divergence (Geometric Interpretation)

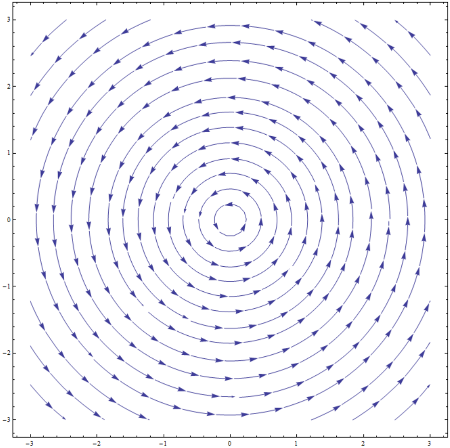
$$\vec{\mathbf{F}}(x,y) = \langle 3, 2 \rangle$$



$$\nabla \cdot \vec{\mathbf{F}} = \frac{\partial}{\partial x}\Big[3\Big] + \frac{\partial}{\partial y}\Big[2\Big] = 0 + 0 = 0$$

Zero Divergence (Geometric Interpretation)

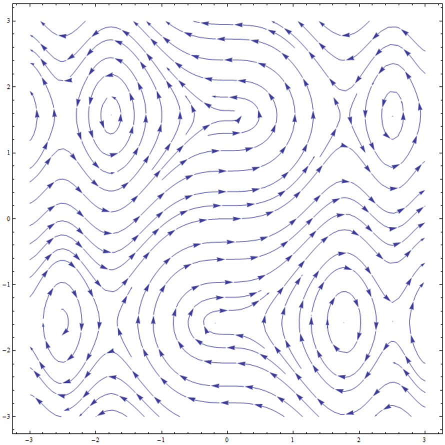
$$\vec{\mathbf{F}}(x,y) = \langle -y, x \rangle$$



$$\nabla \cdot \vec{\mathbf{F}} = \frac{\partial}{\partial x}\Big[-y\Big] + \frac{\partial}{\partial y}\Big[x\Big] = 0 + 0 = 0$$

Zero Divergence (Geometric Interpretation)

$$\vec{\mathbf{F}}(x,y) = \langle \cos y, \sin\left(x^2\right)\rangle$$



$$\nabla \cdot \vec{\mathbf{F}} = \frac{\partial}{\partial x}\Big[\cos y\Big] + \frac{\partial}{\partial y}\Big[\sin\left(x^2\right)\Big] = 0 + 0 = 0$$

Curl of a Vector Field

Definition

Let 3D vector field $\mathbf{F}(x,y,z) \in C^{(1,1,1)}$ s.t.
$\mathbf{F}(x,y,z) = \langle M(x,y,z), N(x,y,z), P(x,y,z)\rangle$.
Then the **curl** of $\mathbf{F}(x,y,z)$ is

$$\text{curl } \mathbf{F} \equiv \nabla \times \mathbf{F} := \begin{vmatrix} \widehat{\mathbf{i}} & \widehat{\mathbf{j}} & \widehat{\mathbf{k}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ M & N & P \end{vmatrix} = \left\langle \frac{\partial P}{\partial y} - \frac{\partial N}{\partial z}, \frac{\partial M}{\partial z} - \frac{\partial P}{\partial x}, \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right\rangle$$

REMARK: The curl of a vector field is a vector field.

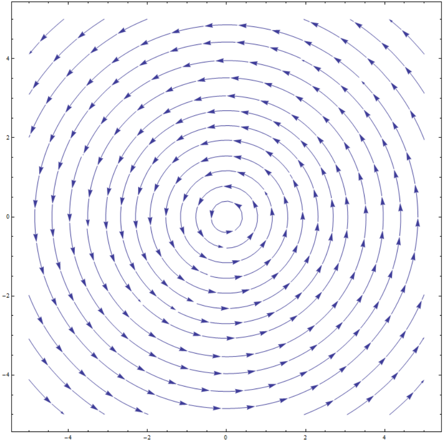
REMARK: Treat 2D vector fields as 3D vector fields with 3^{rd} component zero:

$$\mathbf{F}(x,y) = \langle M(x,y), N(x,y)\rangle \xrightarrow{extends\ to} \overline{\mathbf{F}}(x,y,z) = \langle M(x,y), N(x,y), 0\rangle$$

$$\text{Then, curl } \mathbf{F} = \text{ curl } \overline{\mathbf{F}} = \left\langle 0, 0, \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right\rangle = \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) \widehat{\mathbf{k}}$$

Curl (Geometric Interpretation)

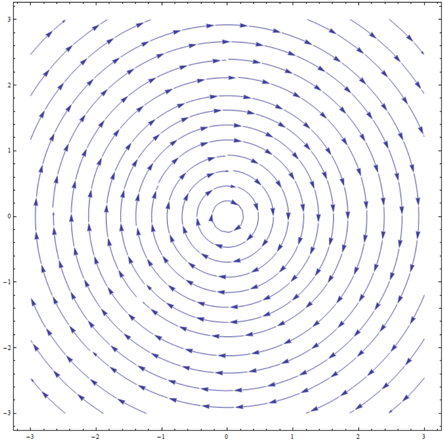
$$\vec{\mathbf{F}}(x,y) = \langle -y, x \rangle$$



$$\nabla \times \vec{\mathbf{F}} = \left\langle 0, 0, \frac{\partial}{\partial x}\Big[x\Big] - \frac{\partial}{\partial y}\Big[-y\Big] \right\rangle = \langle 0, 0, 1-(-1) \rangle = \langle 0, 0, 2 \rangle$$

Curl (Geometric Interpretation)

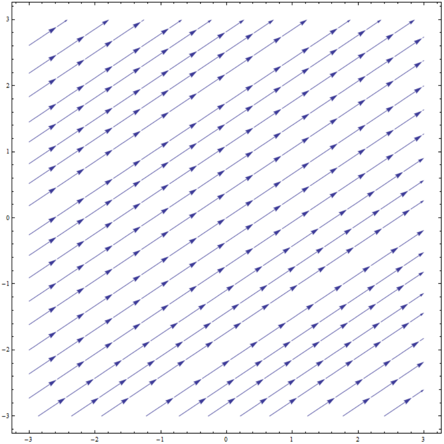
$$\vec{\mathbf{F}}(x, y) = \langle y, -x \rangle$$



$$\nabla \times \vec{\mathbf{F}} = \left\langle 0, 0, \frac{\partial}{\partial x}\Big[-x\Big] - \frac{\partial}{\partial y}\Big[y\Big] \right\rangle = \langle 0, 0, -1 - 1 \rangle = \langle 0, 0, -2 \rangle$$

Curl (Geometric Interpretation)

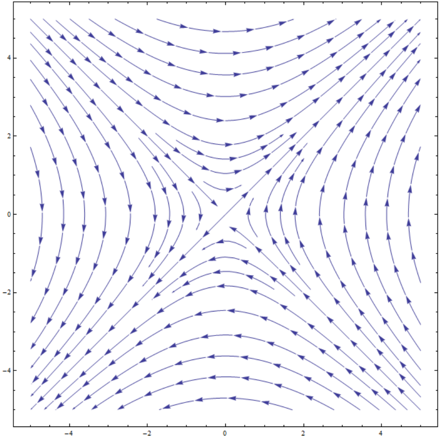
$$\vec{\mathbf{F}}(x, y) = \langle 3, 2 \rangle$$



$$\nabla \times \vec{\mathbf{F}} = \left\langle 0, 0, \frac{\partial}{\partial x}\Big[2\Big] - \frac{\partial}{\partial y}\Big[3\Big] \right\rangle = \langle 0, 0, 0 - 0 \rangle = \langle 0, 0, 0 \rangle = \vec{\mathbf{0}}$$

Curl (Geometric Interpretation)

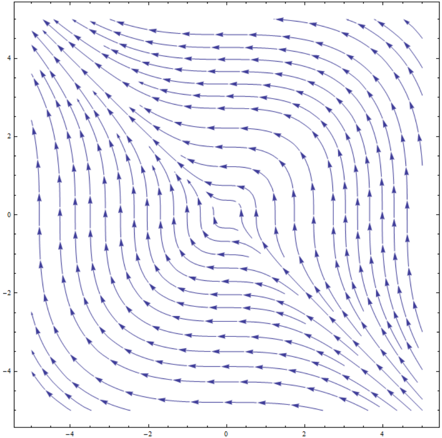
$$\vec{\mathbf{F}}(x,y) = \langle y, x \rangle$$



$$\nabla \times \vec{\mathbf{F}} = \left\langle 0, 0, \frac{\partial}{\partial x}\Big[x\Big] - \frac{\partial}{\partial y}\Big[y\Big] \right\rangle = \langle 0, 0, 1 - 1 \rangle = \langle 0, 0, 0 \rangle = \vec{\mathbf{0}}$$

Curl (Geometric Interpretation)

$$\vec{\mathbf{F}}(x,y) = \left\langle -\frac{1}{2}y^2, \frac{1}{2}x^2 \right\rangle$$



$$\nabla \times \vec{\mathbf{F}} = \left\langle 0, 0, \frac{\partial}{\partial x}\left[\frac{1}{2}x^2\right] - \frac{\partial}{\partial y}\left[-\frac{1}{2}y^2\right] \right\rangle = \langle 0, 0, x - (-y) \rangle = \langle 0, 0, x + y \rangle$$

Laplacian of a Scalar Field

Definition

Let $f(x, y) \in C^{(2,2)}$. Then the **Laplacian** of f is

$$\nabla^2 f := \text{div}\left(\text{grad}\, f\right) \equiv \nabla \cdot \nabla f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = f_{xx} + f_{yy}$$

Definition

Let $f(x, y, z) \in C^{(2,2,2)}$. Then the **Laplacian** of f is

$$\nabla^2 f := \text{div}\left(\text{grad}\, f\right) \equiv \nabla \cdot \nabla f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} = f_{xx} + f_{yy} + f_{zz}$$

WEX 13-1-2: Compute the Laplacian of $f(x, y) = x^4 + y^3$.

$f_x = 4x^3 \qquad f_y = 3y^2$

$f_{xx} = 12x^2 \qquad f_{yy} = 6y$

$$\therefore \nabla^2 f = f_{xx} + f_{yy} = \boxed{12x^2 + 6y}$$

Harmonic Functions

Definition

Let $D \subset \mathbb{R}^2$ be a region on the xy-plane.
Let function $f(x,y) \in C^{(2,2)}(D)$. Then

f is **harmonic** in D $\iff$ $f \in \text{Har}(D)$ $\iff$ $\nabla^2 f = 0$ $\iff$ $\dfrac{\partial^2 f}{\partial x^2} + \dfrac{\partial^2 f}{\partial y^2} = 0$

Definition

Let $E \subset \mathbb{R}^3$ be a solid in xyz-space.
Let function $f(x,y,z) \in C^{(2,2,2)}(E)$. Then

f is **harmonic** in E $\iff$ $f \in \text{Har}(E)$ $\iff$ $\nabla^2 f = 0$ $\iff$ $\dfrac{\partial^2 f}{\partial x^2} + \dfrac{\partial^2 f}{\partial y^2} + \dfrac{\partial^2 f}{\partial z^2} = 0$

WEX 13-1-3: Is $f(x,y) = x^2 - y^2$ harmonic in $\mathbb{R}^2$? (Justify answer)

$f_x = 2x \qquad f_y = -2y \qquad f_{xx} = 2 \qquad f_{yy} = -2$

$\implies \nabla^2 f = f_{xx} + f_{yy} = 2 + (-2) = 0 \quad \forall (x,y) \in \mathbb{R}^2$

$\therefore$ Since $\nabla^2 f = 0$, f **is** harmonic in $\mathbb{R}^2$

Summary of Operations on Vector Fields

OPERATION	DEL OPERATOR NOTATION	ALTERNATIVE NOTATION
Gradient	∇f	grad f
Divergence	$\nabla \cdot \mathbf{F}$	div $\mathbf{F}$
Curl	$\nabla \times \mathbf{F}$	curl $\mathbf{F}$
Laplacian	$\nabla^2 f$	div $\left(\text{grad } f\right)$

Vector Field, Scalar	$\to$	$k\mathbf{F}$	$\to$	Vector Field
2 Vector Fields	$\to$	$\mathbf{F} + \mathbf{G}$	$\to$	Vector Field
2 Vector Fields	$\to$	$\mathbf{F} - \mathbf{G}$	$\to$	Vector Field
2 Vector Fields	$\to$	$\mathbf{F} \cdot \mathbf{G}$	$\to$	Scalar Field
2 Vector Fields	$\to$	$\mathbf{F} \times \mathbf{G}$	$\to$	Vector Field
Vector Field, Scalar Field	$\to$	$f\mathbf{F}$	$\to$	Vector Field
Scalar Field	$\to$	∇f	$\to$	Vector Field
Scalar Field	$\to$	$\nabla^2 f$	$\to$	Scalar Field
Vector Field	$\to$	$\nabla \cdot \mathbf{F}$	$\to$	Scalar Field
Vector Field	$\to$	$\nabla \times \mathbf{F}$	$\to$	Vector Field

Div & Curl (Properties & Identities)

$$\left(k \in \mathbb{R} \qquad f, g : \mathbb{R}^3 \to \mathbb{R} \qquad \mathbf{F}, \mathbf{G} : \mathbb{R}^3 \to \mathbb{R}^3\right)$$

$\text{div}(k\mathbf{F}) = k \text{ div } \mathbf{F}$	$\text{curl}(k\mathbf{F}) = k \text{ curl } \mathbf{F}$
$\text{div}(\mathbf{F} \pm \mathbf{G}) = \text{ div } \mathbf{F} \pm \text{ div } \mathbf{G}$	$\text{curl}(\mathbf{F} \pm \mathbf{G}) = \text{ curl } \mathbf{F} \pm \text{ curl } \mathbf{G}$
$\text{div}(f\mathbf{F}) = f \text{ div } \mathbf{F} + (\nabla f \cdot \mathbf{F})$	$\text{curl}(f\mathbf{F}) = f \text{ curl } \mathbf{F} + (\nabla f \times \mathbf{F})$
$\text{div}(\text{curl } \mathbf{F}) = 0$	$\text{curl}(\nabla f) = \vec{\mathbf{0}}$

$$\text{div}(f\nabla g) = f \text{ div } (\nabla g) + \nabla f \cdot \nabla g$$
$$\text{div}(\mathbf{F} \times \mathbf{G}) = \text{curl } \mathbf{F} \cdot \mathbf{G} - \mathbf{F} \cdot \text{curl } \mathbf{G}$$
$$\nabla^2(fg) = f\nabla^2 g + 2\nabla f \cdot \nabla g + g\nabla^2 f$$

Vector Calculus Identities NOT to be considered:

- $\nabla(\mathbf{F} \cdot \mathbf{G})$
- $\text{curl}(\mathbf{F} \times \mathbf{G})$
- $\text{curl}(\text{curl } \mathbf{F})$
- Any Identity involving terms of the forms $(\mathbf{G} \cdot \nabla)\mathbf{F}$ or $(\mathbf{G} \cdot \nabla)f$

Div & Curl (Properties & Identities)

$$\left(k \in \mathbb{R} \qquad f, g : \mathbb{R}^3 \to \mathbb{R} \qquad \mathbf{F}, \mathbf{G} : \mathbb{R}^3 \to \mathbb{R}^3\right)$$

$\nabla \cdot (k\mathbf{F}) = k\nabla \cdot \mathbf{F}$	$\nabla \times (k\mathbf{F}) = k\nabla \times \mathbf{F}$
$\nabla \cdot (\mathbf{F} \pm \mathbf{G}) = \nabla \cdot \mathbf{F} \pm \nabla \cdot \mathbf{G}$	$\nabla \times (\mathbf{F} \pm \mathbf{G}) = \nabla \times \mathbf{F} \pm \nabla \times \mathbf{G}$
$\nabla \cdot (f\mathbf{F}) = f(\nabla \cdot \mathbf{F}) + (\nabla f \cdot \mathbf{F})$	$\nabla \times (f\mathbf{F}) = f(\nabla \times \mathbf{F}) + (\nabla f \times \mathbf{F})$
$\nabla \cdot (\nabla \times \mathbf{F}) = 0$	$\nabla \times (\nabla f) = \vec{\mathbf{0}}$

$$\nabla \cdot (f\nabla g) = f\nabla \cdot (\nabla g) + \nabla f \cdot \nabla g$$
$$\nabla \cdot (\mathbf{F} \times \mathbf{G}) = (\nabla \times \mathbf{F}) \cdot \mathbf{G} - \mathbf{F} \cdot (\nabla \times \mathbf{G})$$
$$\nabla^2(fg) = f\nabla^2 g + 2\nabla f \cdot \nabla g + g\nabla^2 f$$

Vector Calculus Identities NOT to be considered:

- $\nabla(\mathbf{F} \cdot \mathbf{G})$
- $\nabla \times (\mathbf{F} \times \mathbf{G})$
- $\nabla \times (\nabla \times \mathbf{F})$
- Any Identity involving terms of the forms $(\mathbf{G} \cdot \nabla)\mathbf{F}$ or $(\mathbf{G} \cdot \nabla)f$

Proof that $\nabla \cdot (\nabla \times \mathbf{F}) = 0$

Let vector field $\mathbf{F} \in C^{(2,2,2)}$ s.t. $\mathbf{F}(x,y,z) = \langle M(x,y,z), N(x,y,z), P(x,y,z)\rangle$.

Prove: $\nabla \cdot (\nabla \times \mathbf{F}) = 0$

PROOF:

$$\begin{aligned}
\nabla \cdot (\nabla \times \mathbf{F}) &= \left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right\rangle \cdot \left\langle \frac{\partial P}{\partial y} - \frac{\partial N}{\partial z}, \frac{\partial M}{\partial z} - \frac{\partial P}{\partial x}, \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right\rangle \\
&= \frac{\partial}{\partial x}\left[\frac{\partial P}{\partial y} - \frac{\partial N}{\partial z}\right] + \frac{\partial}{\partial y}\left[\frac{\partial M}{\partial z} - \frac{\partial P}{\partial x}\right] + \frac{\partial}{\partial z}\left[\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right] \\
&= \left(\frac{\partial^2 P}{\partial x \partial y} - \frac{\partial^2 N}{\partial x \partial z}\right) + \left(\frac{\partial^2 M}{\partial y \partial z} - \frac{\partial^2 P}{\partial y \partial x}\right) + \left(\frac{\partial^2 N}{\partial z \partial x} - \frac{\partial^2 M}{\partial z \partial y}\right) \\
&= \left(\frac{\partial^2 P}{\partial x \partial y} - \frac{\partial^2 P}{\partial y \partial x}\right) + \left(\frac{\partial^2 N}{\partial z \partial x} - \frac{\partial^2 N}{\partial x \partial z}\right) + \left(\frac{\partial^2 M}{\partial y \partial z} - \frac{\partial^2 M}{\partial z \partial y}\right) \\
&= 0 + 0 + 0 \qquad \left[\text{Since } \mathbf{F} \in C^{(2,2,2)}\text{, mixed } 2^{nd} \text{ partials are equal}\right] \\
&= 0
\end{aligned}$$

QED

Proof that $\nabla \times (\nabla f) = \vec{\mathbf{0}}$

Let scalar field $f \in C^{(2,2,2)}$.

Prove: $\nabla \times (\nabla f) = \vec{\mathbf{0}}$

PROOF:

$$
\begin{aligned}
\nabla \times (\nabla f) &= \left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right\rangle \times \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right\rangle = \begin{vmatrix} \widehat{\mathbf{i}} & \widehat{\mathbf{j}} & \widehat{\mathbf{k}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} & \frac{\partial f}{\partial z} \end{vmatrix} \\
&= \begin{vmatrix} \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{\partial f}{\partial y} & \frac{\partial f}{\partial z} \end{vmatrix} \widehat{\mathbf{i}} - \begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial z} \\ \frac{\partial f}{\partial x} & \frac{\partial f}{\partial z} \end{vmatrix} \widehat{\mathbf{j}} + \begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} \\ \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \end{vmatrix} \widehat{\mathbf{k}} \\
&= \left\langle \frac{\partial^2 f}{\partial y \partial z} - \frac{\partial^2 f}{\partial z \partial y}, \frac{\partial^2 f}{\partial z \partial x} - \frac{\partial^2 f}{\partial x \partial z}, \frac{\partial^2 f}{\partial x \partial y} - \frac{\partial^2 f}{\partial y \partial x} \right\rangle \\
&= \langle 0, 0, 0 \rangle \qquad \left[\text{Since } f \in C^{(2,2,2)}, \text{ mixed } 2^{nd} \text{ partials are equal}\right] \\
&= \vec{\mathbf{0}}
\end{aligned}
$$

QED

Not every expression involving div, grad, curl makes sense!

WEX 13-1-4: Explain why $\nabla \times (x^2 + yz)$ makes no sense.

$x^2 + yz$ is a scalar field, but the curl of a scalar field is not defined.

WEX 13-1-5: Let $\mathbf{F}$ be a vector field.
Explain why $\nabla \cdot (\nabla \cdot \mathbf{F})$ makes no sense.

$\nabla \cdot \mathbf{F}$ yields a scalar field, but the divergence of a scalar field is not defined.

WEX 13-1-6: Let f be a scalar field & $\mathbf{F}$ be a vector field.
Explain why $(\nabla f) \times (\nabla \cdot \mathbf{F})$ makes no sense.

∇f yields a vector field,
$\nabla \cdot \mathbf{F}$ yields a scalar field,
but the cross product of a scalar field and a vector field is not defined.

Fin.