Line Integrals

Calculus III

Josh Engwer

TTU

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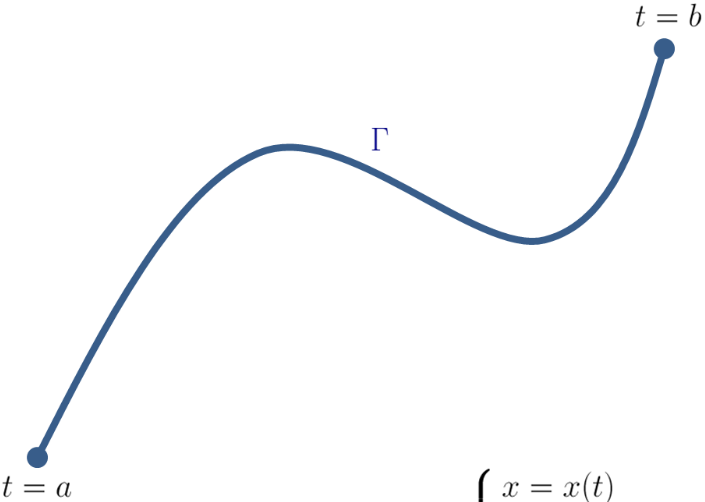
PART I:

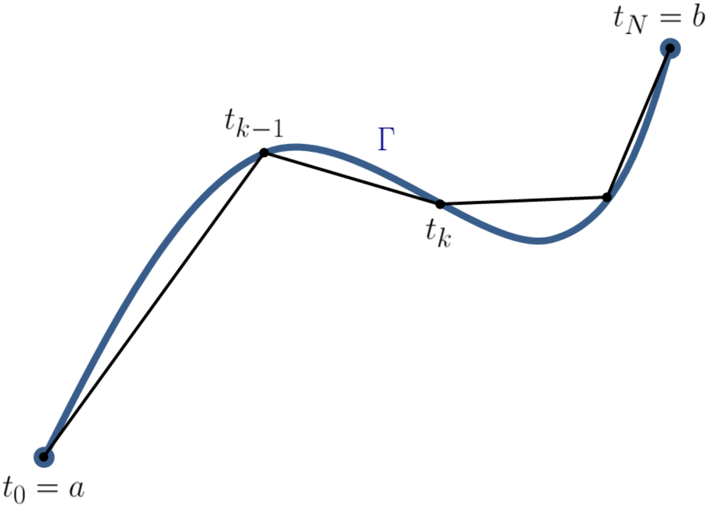
LINE INTEGRALS OF SCALAR FIELDS

Line Integral of a Scalar Field f in $\mathbb{R}^2$ (Derivation)



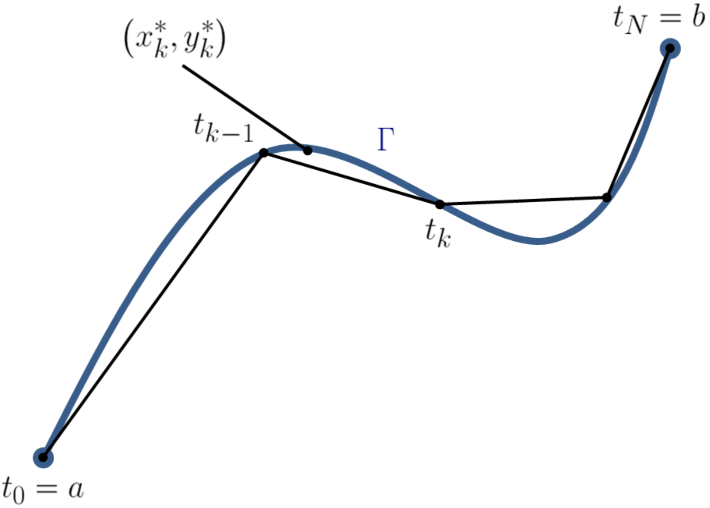
$$\text{Given smooth curve } \Gamma : \begin{cases} x = x(t) \\ y = y(t) \\ t \in [a, b] \end{cases}$$

Line Integral of a Scalar Field f in $\mathbb{R}^2$ (Derivation)



Partition curve Γ into N subarcs & line segments

Line Integral of a Scalar Field f in $\mathbb{R}^2$ (Derivation)



Pick a point (x_k^*, y_k^*) on the k^{th} subarc

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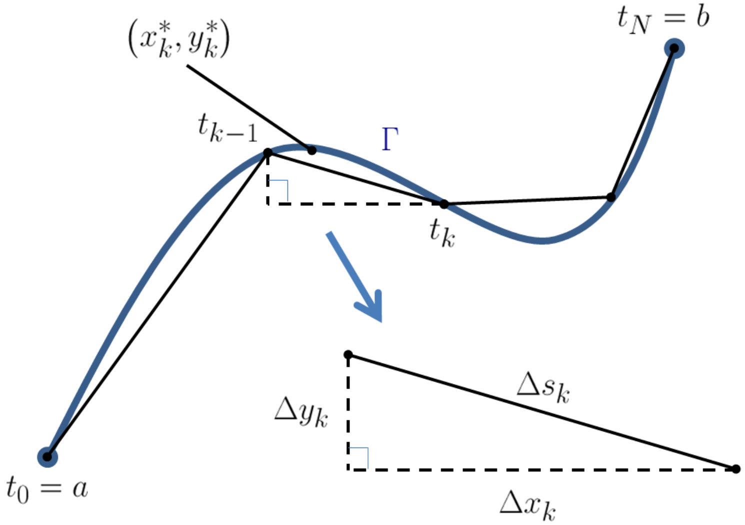
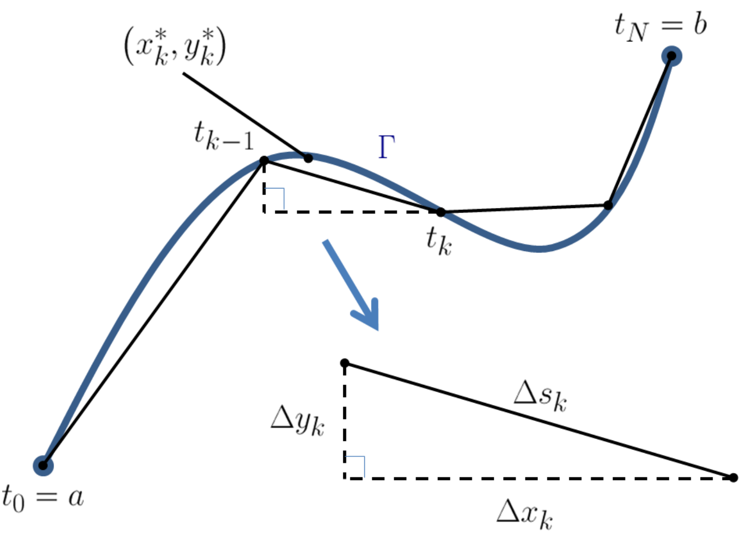


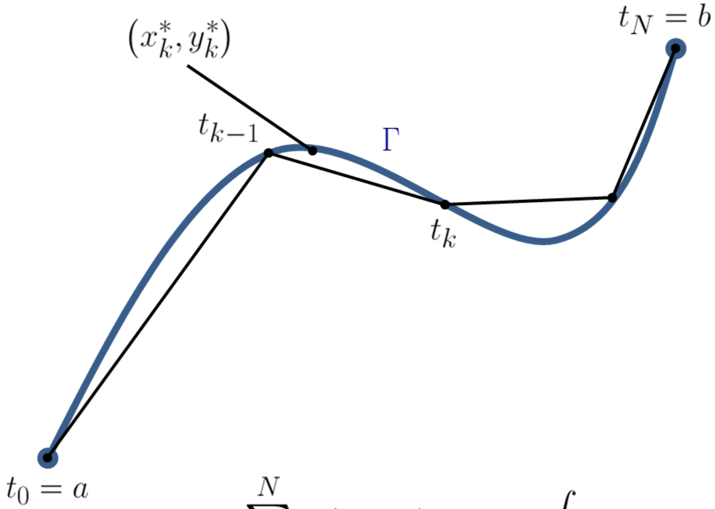
Label the sides of the k^{th} right triangle

Line Integral of a Scalar Field f in $\mathbb{R}^2$ (Derivation)



$$\Delta s_k = \sqrt{(\Delta x_k)^2 + (\Delta y_k)^2}$$

Line Integral of a Scalar Field f in $\mathbb{R}^2$ (Derivation)



$$\lim_{N\to\infty} \sum_{k=1}^{N} f\left(x_k^*, y_k^*\right) \Delta s_k := \int_\Gamma f(x, y)\, ds$$

Line Integral of a Scalar Field f in $\mathbb{R}^2$ (Derivation)

$$\Delta s_k = \sqrt{(\Delta x_k)^2 + (\Delta y_k)^2}$$
$$\implies ds = \sqrt{(dx)^2 + (dy)^2} \qquad \Big(\text{After taking limits on both sides}\Big)$$

Now, the parameterization of Γ : $\begin{cases} x = x(t) \\ y = y(t) \\ t \in [a, b] \end{cases}$

$$\implies \frac{dx}{dt} = x'(t) \implies dx = x'(t)\ dt$$
$$\implies \frac{dy}{dt} = y'(t) \implies dy = y'(t)\ dt$$

$$\therefore ds = \sqrt{[x'(t)]^2 + [y'(t)]^2}\ dt$$

$$\therefore \underbrace{\int_\Gamma f(x,y)\ ds}_{\text{Line Integral}} = \underbrace{\int_a^b f\Big[x(t), y(t)\Big]\sqrt{[x'(t)]^2 + [y'(t)]^2}\ dt}_{\text{Ordinary Single Integral}}$$

Line Integral of a Scalar Field in $\mathbb{R}^2$ (Definition)

Definition

Given smooth curve $\Gamma : \begin{cases} x = x(t) \\ y = y(t) \\ t \in [a, b] \end{cases}$ and scalar field $f(x, y) \in C(\Gamma)$

Then the **line integral** of f over curve Γ is defined to be

$$\int_\Gamma f(x, y)\, ds := \int_a^b f\Big[x(t), y(t)\Big] \sqrt{[x'(t)]^2 + [y'(t)]^2}\, dt$$

REMARK: Very often, the curve is denoted by C:

$$\int_C f(x, y)\, ds := \int_a^b f\Big[x(t), y(t)\Big] \sqrt{[x'(t)]^2 + [y'(t)]^2}\, dt$$

Line Integral of a Scalar Field in $\mathbb{R}^3$ (Definition)

Definition

Given smooth curve $\Gamma : \begin{cases} x = x(t) \\ y = y(t) \\ z = z(t) \\ t \in [a,b] \end{cases}$ and scalar field $f(x,y,z) \in C(\Gamma)$.

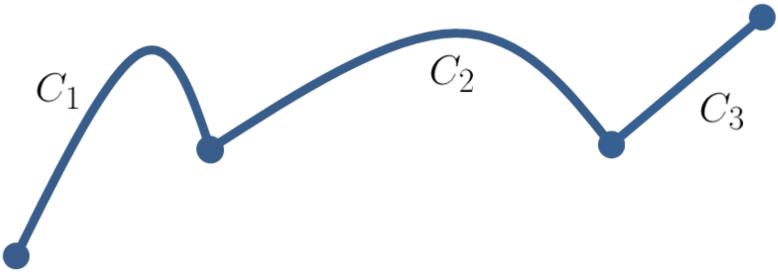
Then the **line integral** of f over curve Γ is defined to be

$$\int_\Gamma f(x,y,z)\ ds := \int_a^b f\Big[x(t), y(t), z(t)\Big] \sqrt{[x'(t)]^2 + [y'(t)]^2 + [z'(t)]^2}\ dt$$

REMARK: Very often, the curve is denoted by C:

$$\int_C f(x,y,z)\ ds := \int_a^b f\Big[x(t), y(t), z(t)\Big] \sqrt{[x'(t)]^2 + [y'(t)]^2 + [z'(t)]^2}\ dt$$

Line Integral over a Piecewise Smooth Curve



$$C = C_1 \cup C_2 \cup C_3 \implies \int_C f\, ds = \int_{C_1} f\, ds + \int_{C_2} f\, ds + \int_{C_3} f\, ds$$

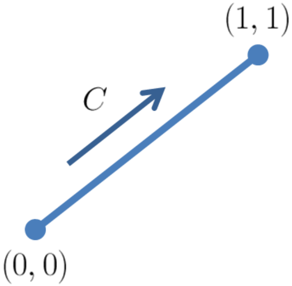
Parameterizing Curves

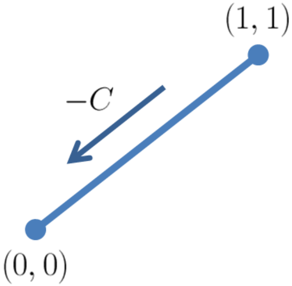
TASK: Parameterize a curve starting at $t = a$ and ending at $t = b$.

CURVE	PARAMETERIZATION
$y = f(x)$	$\begin{cases} x = t \\ y = f(t) \\ t \in [a, b] \end{cases}$
$x = g(y)$	$\begin{cases} x = g(t) \\ y = t \\ t \in [a, b] \end{cases}$
$x^2 + y^2 = r^2$	$\begin{cases} x = r\cos t \\ y = r\sin t \\ t \in [a, b] \end{cases}$
$\dfrac{x^2}{A^2} + \dfrac{y^2}{B^2} = 1$	$\begin{cases} x = A\cos t \\ y = B\sin t \\ t \in [a, b] \end{cases}$
$\dfrac{x^2}{A^2} - \dfrac{y^2}{B^2} = 1$	$\begin{cases} x = A\sec t \\ y = B\tan t \\ t \in [a, b] \end{cases}$

REMARK: Parameterizations are NOT unique.

Line Integral (Curve Orientation)





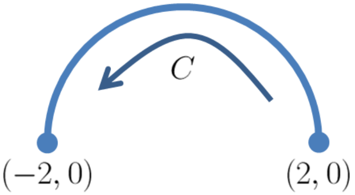
$$C : \begin{cases} x(t) = t \\ y(t) = t \\ t \in [0,1] \end{cases} \qquad -C : \begin{cases} x(t) = 1 - t \\ y(t) = 1 - t \\ t \in [0,1] \end{cases}$$

$$t = 0 \rightarrow \text{point } (0,0) \qquad t = 0 \rightarrow \text{point } (1,1)$$

$$t = 1 \rightarrow \text{point } (1,1) \qquad t = 1 \rightarrow \text{point } (0,0)$$

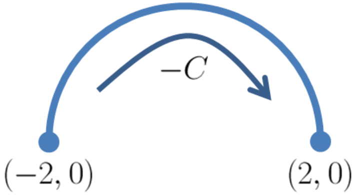
$$\int_{-C} f \, ds = -\int_{C} f \, ds$$

Line Integral (Curve Orientation)



$$C : \begin{cases} x(t) = 2\cos t \\ y(t) = 2\sin t \\ t \in [0, \pi] \end{cases}$$

$$t = 0 \to \text{point } (2, 0)$$
$$t = \pi \to \text{point } (-2, 0)$$

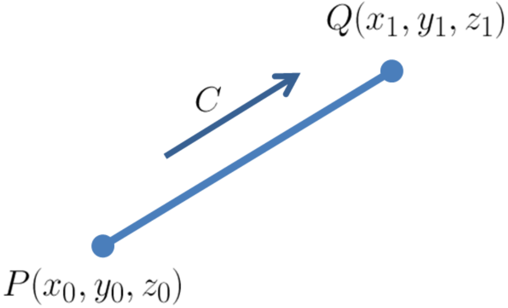


$$-C : \begin{cases} x(t) = -2\cos t \\ y(t) = 2\sin t \\ t \in [0, \pi] \end{cases}$$

$$t = 0 \to \text{point } (-2, 0)$$
$$t = \pi \to \text{point } (2, 0)$$

$$\int_{-C} f \; ds = -\int_{C} f \; ds$$

Parameterizing Line Segments



$$C : \begin{cases} x = (1-t)x_0 + tx_1 \\ y = (1-t)y_0 + ty_1 \\ z = (1-t)z_0 + tz_1 \\ t \in [0,1] \end{cases} \iff C : \begin{cases} x = x_0 + (x_1 - x_0)t \\ y = y_0 + (y_1 - y_0)t \\ z = z_0 + (z_1 - z_0)t \\ t \in [0,1] \end{cases}$$

Line Integral of a Scalar Field f in $\mathbb{R}^2$ w.r.t. x, y

Proposition

Given smooth curve $C : \begin{cases} x = x(t) \\ y = y(t) \\ t \in [a, b] \end{cases}$ *and continuous scalar field* $f(x, y)$.

Then:

$$\int_C f(x,y)\ dx = \int_a^b f\Big[x(t), y(t)\Big]\ x'(t)\ dt$$

$$\int_C f(x,y)\ dy = \int_a^b f\Big[x(t), y(t)\Big]\ y'(t)\ dt$$

Line Integral of a Scalar Field f in $\mathbb{R}^3$ w.r.t. x, y, z

Proposition

Given smooth curve $C : \begin{cases} x = x(t) \\ y = y(t) \\ z = z(t) \\ t \in [a,b] \end{cases}$ *and continuous scalar field* $f(x,y,z)$.

Then:

$$\int_C f(x,y,z)\ dx = \int_a^b f\Big[x(t), y(t), z(t)\Big]\ x'(t)\ dt$$

$$\int_C f(x,y,z)\ dy = \int_a^b f\Big[x(t), y(t), z(t)\Big]\ y'(t)\ dt$$

$$\int_C f(x,y,z)\ dz = \int_a^b f\Big[x(t), y(t), z(t)\Big]\ z'(t)\ dt$$

Line Integrals (Thin Wires)

Proposition

(Center of Mass of a Thin Wire)

Given smooth curve $C : \{x = x(t),\ \ y = y(t),\ \ z = z(t),\ \ t \in [a, b]\}$

Let a thin wire W *have the shape of curve* C*.*
Suppose the wire has mass-density $\rho(x, y, z)$ *at each point* (x, y, z) *on the wire.*

Then, the ***center of mass*** *of* W *is* $(\bar{x}, \bar{y}, \bar{z})$*, where:*

$$\text{Mass}(W) = \int_C \rho(x, y, z)\, ds$$

$$\bar{x} = \frac{1}{\text{Mass}(W)} \int_C x\rho(x, y, z)\, ds$$

$$\bar{y} = \frac{1}{\text{Mass}(W)} \int_C y\rho(x, y, z)\, ds$$

$$\bar{z} = \frac{1}{\text{Mass}(W)} \int_C z\rho(x, y, z)\, ds$$

REMARK: The center of mass may not lie on the wire itself!

PART II:

LINE INTEGRALS OF VECTOR FIELDS

Line Integral of a Vector Field $\vec{\mathbf{F}}$ in $\mathbb{R}^2$

Definition

Let vector function $\vec{\mathbf{R}}(t) = \langle x(t), y(t) \rangle$ trace out a smooth curve C in $\mathbb{R}^2$.
Let vector field $\vec{\mathbf{F}}(x, y) = \langle M(x, y), N(x, y) \rangle$ be continuous on C.

Then the **line integral of $\vec{\mathbf{F}}$ along** C is defined by

$$\int_C \vec{\mathbf{F}} \cdot d\vec{\mathbf{R}} := \int_C \Big[M(x, y)\ dx + N(x, y)\ dy \Big]$$

where $d\vec{\mathbf{R}} = \langle dx, dy \rangle = \langle x'(t), y'(t) \rangle\ dt$

Alternatively,

$$\int_C \vec{\mathbf{F}} \cdot d\vec{\mathbf{R}} = \int_a^b \vec{\mathbf{F}}\Big[\vec{\mathbf{R}}(t)\Big] \cdot \vec{\mathbf{R}}'(t)\ dt = \int_a^b \Big[M\big(x(t), y(t)\big) x'(t) + N\big(x(t), y(t)\big) y'(t) \Big]\ dt$$

where vector function $\vec{\mathbf{F}}\Big[\vec{\mathbf{R}}(t)\Big] = \Big\langle M\big(x(t), y(t)\big), N\big(x(t), y(t)\big) \Big\rangle$

Line Integral of a Vector Field $\vec{\mathbf{F}}$ in $\mathbb{R}^3$

Definition

Let vector function $\vec{\mathbf{R}}(t) = \langle x(t), y(t), z(t) \rangle$ trace out a smooth curve C in $\mathbb{R}^3$.
Let vector field $\vec{\mathbf{F}}(x,y,z) = \langle M(x,y,z), N(x,y,z), P(x,y,z) \rangle$ be continuous on C.

Then the **line integral of $\vec{\mathbf{F}}$ along** C is defined by

$$\int_C \vec{\mathbf{F}} \cdot d\vec{\mathbf{R}} := \int_C \Big[M(x,y,z)\ dx + N(x,y,z)\ dy + P(x,y,z)\ dz \Big]$$

where $d\vec{\mathbf{R}} = \langle dx, dy, dz \rangle = \langle x'(t), y'(t), z'(t) \rangle\ dt$

Line Integrals in $\mathbb{R}^2$ (Work)

Recall from SST 9.3 that **dot products** allowed us to compute the **work** done by a **constant force** acting on an object along a **line**:

$$\text{Work} = \vec{\mathbf{F}} \cdot \mathbf{PQ}$$

Line integrals allows us to compute the **work** done by a **force field** (i.e. **variable force**) acting on an object along a **curve**:

Proposition

*Let vector function $\vec{\mathbf{R}}(t) = \langle x(t), y(t) \rangle$ trace out a smooth curve C in $\mathbb{R}^2$. Let **force field** $\vec{\mathbf{F}}(x, y)$ be continuous on C.*

*Then the **work** done by $\vec{\mathbf{F}}$ on an object moving it along the curve C is*

$$\textit{Work} = \int_C \vec{\mathbf{F}} \cdot d\vec{\mathbf{R}}$$

Line Integrals in $\mathbb{R}^3$ (Work)

Recall from SST 9.3 that **dot products** allowed us to compute the **work** done by a **constant force** acting on an object along a **line**:

$$\text{Work} = \vec{\mathbf{F}} \cdot \mathbf{PQ}$$

Line integrals allows us to compute the **work** done by a **force field** (i.e. **variable force**) acting on an object along a **curve**:

Proposition

Let vector function $\vec{\mathbf{R}}(t) = \langle x(t), y(t), z(t) \rangle$ *trace out a smooth curve* C *in* $\mathbb{R}^3$*.*
*Let **force field*** $\vec{\mathbf{F}}(x,y,z)$ *be continuous on* C*.*

*Then the **work** done by* $\vec{\mathbf{F}}$ *on an object moving it along the curve* C *is*

$$\textit{Work} = \int_C \vec{\mathbf{F}} \cdot d\vec{\mathbf{R}}$$

Fin.