Gradient Fields & Scalar Potentials

Calculus III

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TTU

17 November 2014

PART I:

GRADIENT FIELDS & SCALAR POTENTIALS

Gradient Fields & Scalar Potentials (Definition)

Definition

(Gradient Field in $\mathbb{R}^2$)

Let $D \subset \mathbb{R}^2$ be a region on the xy-plane. Then

$$\vec{\mathbf{F}} \text{ is a } \textbf{gradient field} \iff \vec{\mathbf{F}}(x,y) = \nabla f(x,y) \quad \forall (x,y) \in D$$

for some scalar field f which is called the **scalar potential** of $\vec{\mathbf{F}}$ in D.

Definition

(Gradient Field in $\mathbb{R}^3$)

Let $E \subset \mathbb{R}^3$ be a solid in xyz-space. Then

$$\vec{\mathbf{F}} \text{ is a } \textbf{gradient field} \iff \vec{\mathbf{F}}(x,y,z) = \nabla f(x,y,z) \quad \forall (x,y,z) \in E$$

for some scalar field f which is called the **scalar potential** of $\vec{\mathbf{F}}$ in E.

REMARK: Another name for **gradient field** is **conservative vector field**.

Fundamental Theorem for Line Integrals (FTLI)

Recall from Calculus I the **Fundamental Theorem of Calculus (FTC)**:

Theorem

Let $[a,b]$ *be a closed interval traced out by* x *for* $x \in [a,b]$*.*
Let scalar function $f \in C^1[a,b]$*. Then*

$$\int_a^b f'(x)\,dx = f(b) - f(a)$$

Here is the corresponding **Fundamental Theorem for Line Integrals (FTLI)**:

Theorem

Let Γ *be a piecewise smooth curve traced out by* $\vec{\mathbf{R}}(t)$ *for* $t \in [a,b]$*.*
Let scalar field $f \in C^1(\Gamma)$*. Then*

$$\int_\Gamma \nabla f \cdot d\vec{\mathbf{R}} = f\left[\vec{\mathbf{R}}(b)\right] - f\left[\vec{\mathbf{R}}(a)\right]$$

Proof of the FTLI

PROOF: Given $f(x,y)$, let $\vec{\mathbf{R}}(t) = \langle x(t), y(t)\rangle$ & $G(t) = f\left[\vec{\mathbf{R}}(t)\right] = f\left[x(t), y(t)\right]$.

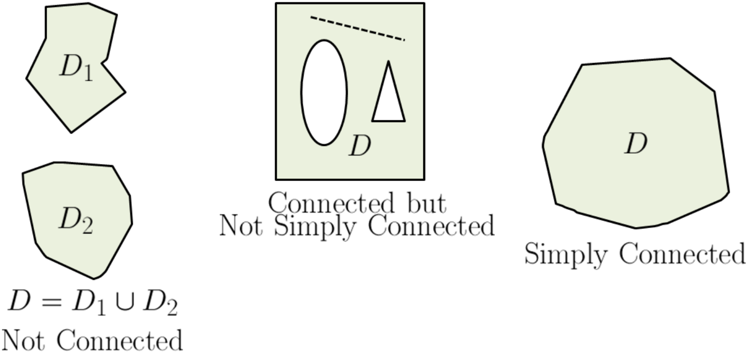
Then $G'(t) = \dfrac{dG}{dt} = \dfrac{\partial f}{\partial x}\dfrac{dx}{dt} + \dfrac{\partial f}{\partial y}\dfrac{dy}{dt}$ (2-1 Chain Rule)

$$\begin{aligned}
\therefore \int_\Gamma \nabla f \cdot d\vec{\mathbf{R}} &= \int_\Gamma \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right\rangle \cdot \langle dx, dy\rangle \\
&= \int_\Gamma \left[\frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy\right] && \text{(Dot Product)} \\
&= \int_a^b \left[\frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt}\right] dt && \left(dx = x'(t)dt = \frac{dx}{dt}dt\right) \\
&= \int_a^b G'(t)\; dt \\
&= G(b) - G(a) && \text{(FTC)} \\
&= f\left[\vec{\mathbf{R}}(b)\right] - f\left[\vec{\mathbf{R}}(a)\right]
\end{aligned}$$

The procedure is similar for $f(x,y,z)$ and $\vec{\mathbf{R}}(t) = \langle x(t), y(t), z(t)\rangle$.

QED

Simply-Connected Regions in $\mathbb{R}^2$



Definition

A **simply-connected set** is a connected set with no holes or cuts.
A **region** is a **connected set** in $\mathbb{R}^2$.
A **solid** is a **connected set** in $\mathbb{R}^3$.

Testing for a Gradient Field

Theorem

(Cross-Partials Test for a Gradient Field in $\mathbb{R}^2$)

Let $D \subseteq \mathbb{R}^2$ be a simply-connected region in the xy-plane. Let vector field $\vec{\mathbf{F}} \in C^{(1,1)}(D)$ s.t. $\vec{\mathbf{F}}(x,y) = \langle M(x,y), N(x,y)\rangle$. Then:

$$\vec{\mathbf{F}} \text{ is conservative} \iff \vec{\mathbf{F}} \text{ is a gradient field} \iff \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Theorem

(Curl Test for a Gradient Field in $\mathbb{R}^3$)

Let $E \subseteq \mathbb{R}^3$ be a simply-connected solid in xyz-space. Let vector field $\vec{\mathbf{F}} \in C^{(1,1,1)}(E)$ s.t. $\vec{\mathbf{F}}(x,y,z) = \langle M(x,y,z), N(x,y,z), P(x,y,z)\rangle$. Then:

$$\vec{\mathbf{F}} \text{ is conservative} \iff \vec{\mathbf{F}} \text{ is a gradient field} \iff \nabla \times \vec{\mathbf{F}} = \vec{\mathbf{0}}$$

PROOF: Requires **Green's Theorem** (which will be covered next time)

Indefinite Integral of a Scalar Field

Recall indefinite integration of a scalar function from Calculus I:

$$\int 3x^2\,dx = x^3 + K \qquad \Big[K \text{ is an arbitrary constant}\Big]$$

For a scalar field of two variables:

$$\int 2xy\,dx = \Big(\text{Treat } y \text{ as constant}\Big) = x^2y + \varphi(y) \qquad \Big[\varphi(y) \text{ is arbitrary function}\Big]$$

$$\int 2xy\,dy = \Big(\text{Treat } x \text{ as constant}\Big) = xy^2 + \varphi(x) \qquad \Big[\varphi(x) \text{ is arbitrary function}\Big]$$

For a scalar field of three variables:

$$\int 2xyz\,dx = \Big(\text{Treat } y \text{ \& } z \text{ as constants}\Big) = x^2yz + \varphi(y,z)$$

$$\int 2xyz\,dy = \Big(\text{Treat } x \text{ \& } z \text{ as constants}\Big) = xy^2z + \varphi(x,z)$$

$$\int 2xyz\,dz = \Big(\text{Treat } x \text{ \& } y \text{ as constants}\Big) = xyz^2 + \varphi(x,y)$$

$$\Big(\varphi \text{ is the Greek letter "phi"}\Big)$$

Scalar Potential Example in $\mathbb{R}^2$ ("1-phi Method")

WEX 13-3-1: Let vector field $\vec{\mathbf{F}}(x,y) = \langle 2x-y, y^2-x\rangle$.
(a) Verify that $\vec{\mathbf{F}}$ is conservative (b) Find a scalar potential f for $\vec{\mathbf{F}}$.

(a) Let $\left\{ \begin{array}{l} M(x,y) = 2x-y \\ N(x,y) = y^2-x \end{array} \right\} \implies \left\{ \begin{array}{l} \partial M/\partial y = \frac{\partial}{\partial y}\left[2x-y\right] = -1 \\ \partial N/\partial x = \frac{\partial}{\partial x}\left[y^2-x\right] = -1 \end{array} \right\}$
Since cross-partials $\partial M/\partial y = \partial N/\partial x$, vector field $\vec{\mathbf{F}}$ is conservative.

(b) $\vec{\mathbf{F}} = \nabla f \implies \langle 2x-y, y^2-x\rangle = \left\langle \dfrac{\partial f}{\partial x}, \dfrac{\partial f}{\partial y} \right\rangle$ $[EQ.1]$

$\implies \dfrac{\partial f}{\partial x} = 2x-y \implies f(x,y) = \displaystyle\int (2x-y)\,dx = x^2 - xy + \varphi(y)$ $[EQ.2]$

Now, $y^2 - x \overset{EQ.1}{=} \dfrac{\partial f}{\partial y} \overset{EQ.2}{=} \dfrac{\partial}{\partial y}\left[x^2 - xy + \varphi(y)\right] = -x + \varphi'(y)$

$\implies \varphi'(y) = y^2 \implies \varphi(y) = \displaystyle\int y^2\,dy = \tfrac{1}{3}y^3 + K$, where K is a **constant**.

$\therefore f(x,y) = x^2 - xy + \frac{1}{3}y^3 + K$, but only 1 scalar potential's needed, so set $K=0$.

$\therefore$ Scalar potential $\boxed{f(x,y) = x^2 - xy + \dfrac{1}{3}y^3}$

Scalar Potential Example in $\mathbb{R}^2$ ("1-phi Method")

<u>WEX 13-3-1:</u> Let vector field $\vec{\mathbf{F}}(x,y) = \langle 2x - y, y^2 - x\rangle$.
(a) Verify that $\vec{\mathbf{F}}$ is conservative (b) Find a scalar potential f for $\vec{\mathbf{F}}$.

(a) Let $\left\{ \begin{array}{l} M(x,y) = 2x - y \\ N(x,y) = y^2 - x \end{array} \right\} \implies \left\{ \begin{array}{l} \partial M/\partial y = \frac{\partial}{\partial y}\left[2x - y\right] = -1 \\ \partial N/\partial x = \frac{\partial}{\partial x}\left[y^2 - x\right] = -1 \end{array} \right\}$

Since cross-partials $\partial M/\partial y = \partial N/\partial x$, vector field $\vec{\mathbf{F}}$ is conservative.

(b) $\vec{\mathbf{F}} = \nabla f \implies \langle 2x - y, y^2 - x\rangle = \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \right\rangle$ $[EQ.1]$

$\implies \frac{\partial f}{\partial y} = y^2 - x \implies f(x,y) = \int \left(y^2 - x\right)\, dy = \frac{1}{3}y^3 - xy + \varphi(x)$ $[EQ.2]$

Now, $2x - y \overset{EQ.1}{=} \frac{\partial f}{\partial x} \overset{EQ.2}{=} \frac{\partial}{\partial x}\left[\frac{1}{3}y^3 - xy + \varphi(x)\right] = -y + \varphi'(x)$

$\implies \varphi'(x) = 2x \implies \varphi(x) = \int 2x\, dx = x^2 + K$, where K is a **constant.**

$\therefore f(x,y) = x^2 - xy + \frac{1}{3}y^3 + K$, but only 1 scalar potential's needed, so set $K = 0$.

$\therefore$ Scalar potential $\boxed{f(x,y) = x^2 - xy + \frac{1}{3}y^3}$

Scalar Potential Example in $\mathbb{R}^2$ ("2-phi Method")

WEX 13-3-1: Let vector field $\vec{\mathbf{F}}(x,y) = \langle 2x - y, y^2 - x \rangle$.

(a) Verify that $\vec{\mathbf{F}}$ is conservative (b) Find a scalar potential f for $\vec{\mathbf{F}}$.

(a) Let $\left\{ \begin{array}{l} M(x,y) = 2x - y \\ N(x,y) = y^2 - x \end{array} \right\} \implies \left\{ \begin{array}{l} \partial M/\partial y = \frac{\partial}{\partial y}\left[2x - y\right] = -1 \\ \partial N/\partial x = \frac{\partial}{\partial x}\left[y^2 - x\right] = -1 \end{array} \right\}$

Since cross-partials $\partial M/\partial y = \partial N/\partial x$, vector field $\vec{\mathbf{F}}$ is conservative.

(b) $\vec{\mathbf{F}} = \nabla f \implies \langle 2x - y, y^2 - x \rangle = \langle \partial f/\partial x, \partial f/\partial y \rangle$

$$\implies \begin{array}{lllll} \frac{\partial f}{\partial x} = 2x - y & \implies & f(x,y) = \int (2x - y)\, dx & = & x^2 - xy + \varphi_1(y) \\ \frac{\partial f}{\partial y} = y^2 - x & \implies & f(x,y) = \int \left(y^2 - x\right)\, dy & = & \frac{1}{3}y^3 - xy + \varphi_2(x) \end{array}$$

$\therefore\ f(x,y) = x^2 - xy + \varphi_1(y) = \frac{1}{3}y^3 - xy + \varphi_2(x)$

Now, by visual inspection of this "chain of equations":

$\implies\ \varphi_1(y) = \frac{1}{3}y^3$ and $\varphi_2(x) = x^2$

$\therefore$ Scalar potential $\boxed{f(x,y) = x^2 - xy + \frac{1}{3}y^3}$

Scalar Potential Example in $\mathbb{R}^3$ ("3-phi Method")

WEX 13-3-2: Given gradient field $\vec{\mathbf{F}}(x,y,z) = \langle yz, xz, xy \rangle$:
Find a scalar potential f for $\vec{\mathbf{F}}$.

$$\vec{\mathbf{F}} = \nabla f \implies \langle yz, xz, xy \rangle = \langle \partial f/\partial x, \partial f/\partial y, \partial f/\partial z \rangle$$

$$\begin{aligned} & \frac{\partial f}{\partial x} = yz \implies f(x,y,z) = \int yz\,dx = xyz + \varphi_1(y,z) \\ \implies & \frac{\partial f}{\partial y} = xz \implies f(x,y,z) = \int xz\,dy = xyz + \varphi_2(x,z) \\ & \frac{\partial f}{\partial z} = xy \implies f(x,y,z) = \int xy\,dz = xyz + \varphi_3(x,y) \end{aligned}$$

$$\therefore\ f(x,y,z) = xyz + \varphi_1(y,z) = xyz + \varphi_2(x,z) = xyz + \varphi_3(x,y)$$

Now, by visual inspection of this "chain of equations":

$\implies \varphi_1(y,z) = \varphi_2(x,z) = \varphi_3(x,y) = K$, where K is a **constant.**

$\implies f(x,y,z) = xyz + K$, but only 1 scalar potential is needed, so let $K = 0$.

$\therefore$ Scalar potential $\boxed{f(x,y,z) = xyz}$

Scalar Potential Example in $\mathbb{R}^3$ ("3-phi Method")

WEX 13-3-3: Given conservative vector field $\vec{\mathbf{F}}(x,y,z) = \langle 2x, 2y, -4z \rangle$:
Find a scalar potential f for $\vec{\mathbf{F}}$.

$$\vec{\mathbf{F}} = \nabla f \implies \langle 2x, 2y, -4z \rangle = \langle \partial f/\partial x, \partial f/\partial y, \partial f/\partial z \rangle$$

$$\begin{aligned}
& \tfrac{\partial f}{\partial x} = 2x && \implies && f(x,y,z) = \int 2x\,dx && = && x^2 + \varphi_1(y,z) \\
\implies \quad & \tfrac{\partial f}{\partial y} = 2y && \implies && f(x,y,z) = \int 2y\,dy && = && y^2 + \varphi_2(x,z) \\
& \tfrac{\partial f}{\partial z} = -4z && \implies && f(x,y,z) = \int -4z\,dz && = && -2z^2 + \varphi_3(x,y)
\end{aligned}$$

$$\therefore\ f(x,y,z) = x^2 + \varphi_1(y,z) = y^2 + \varphi_2(x,z) = -2z^2 + \varphi_3(x,y)$$

Now, by visual inspection of this "chain of equations":

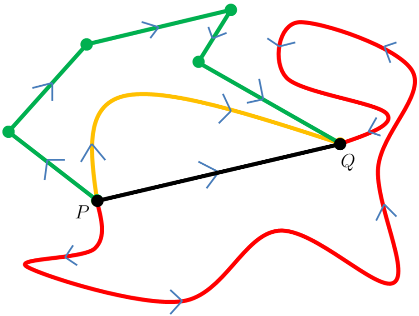
$$\implies f(x,y,z) = x^2 + \big(\text{Fcn of } y,z\big) = y^2 + \big(\text{Fcn of } x,z\big) = -2z^2 + \big(\text{Fcn of } x,y\big)$$
$$\implies f(x,y,z) = \big(\text{Fcn of } x,y,z\big)$$
$$\implies f(x,y,z) = x^2 + y^2 - 2z^2$$

$\therefore$ Scalar potential $\boxed{f(x,y,z) = x^2 + y^2 - 2z^2}$

PART II:

PATH INDEPENDENCE OF CERTAIN LINE INTEGRALS

Path Independence (Definition)

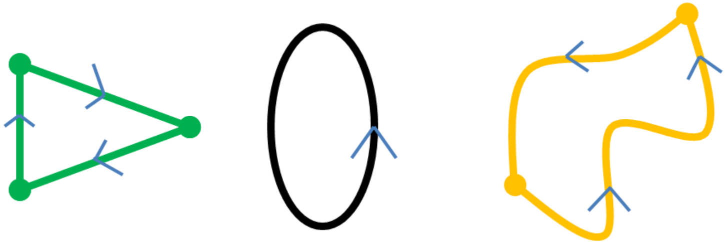


Definition

Let S be an open connected set in either $\mathbb{R}^2$ or $\mathbb{R}^3$.

Then line integral $\int_C \vec{\mathbf{F}} \cdot d\vec{\mathbf{R}}$ is **independent of path** (IoP) in S if for any two points $P, Q \in S$, the line integral along every piecewise smooth curve in S from P to Q has the same value.

Closed Curves



Definition

A **closed curve** in $\mathbb{R}^2$ or $\mathbb{R}^3$ is a curve begins and ends at the same point.

Proposition

Special notation is used with line integrals along ***closed curves****:*

$$\oint_C f\ ds \qquad \oint_C f\ dx \qquad \oint_C f\ dy \qquad \oint_C \vec{\mathbf{F}} \cdot d\vec{\mathbf{R}}$$

Path Independence (Conditions)

Theorem

(Equivalent Conditions for Path Independence)

Let S be an open connected set in either $\mathbb{R}^2$ or $\mathbb{R}^3$.
Let vector field $\vec{\mathbf{F}}$ be continous on S.

Then the following are equivalent (TFAE):

(i) $\vec{\mathbf{F}}$ *is a gradient field on S.*

(ii) $\oint_C \vec{\mathbf{F}} \cdot d\vec{\mathbf{R}} = 0$ *for every piecewise smooth closed curve C in S.*

(iii) $\int_C \vec{\mathbf{F}} \cdot d\vec{\mathbf{R}}$ *is independent of path within S.*

PROOF:

(i) $\Longrightarrow$ (ii): Suppose $\vec{\mathbf{F}}$ is a gradient field. Then $\vec{\mathbf{F}} = \nabla f$ for some scalar field f. Any point P on a closed curve C can serve as both the starting & ending point.

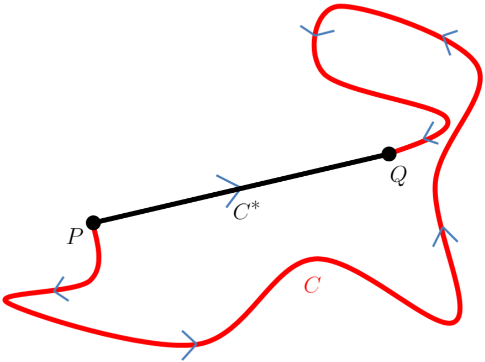
$$\therefore \oint_C \vec{\mathbf{F}} \cdot d\vec{\mathbf{R}} = \oint_C \nabla f \cdot d\vec{\mathbf{R}} \overset{FTLI}{=} f(P) - f(P) = 0$$

(ii) $\Longrightarrow$ (iii): See textbook

(iii) $\Longrightarrow$ (i): See textbook QED

Path Independence (Choosing a Simpler Path)

If finding a scalar potential for $\vec{\mathbf{F}}$ is too tedious or impossible
(due to a nonelementary integral being encountered),
then choose a simple path starting & ending at same points as original path.



$$\underbrace{\int_C \vec{\mathbf{F}} \cdot d\vec{\mathbf{R}}}_{\text{Hard integral}} \overset{IoP}{=} \underbrace{\int_{C^*} \vec{\mathbf{F}} \cdot d\vec{\mathbf{R}}^*}_{\text{Easy integral}}$$

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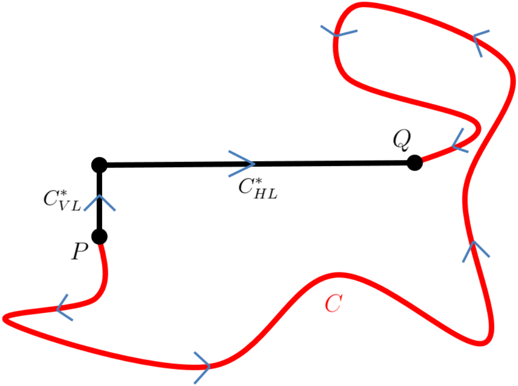

$$\underbrace{\int_C \vec{\mathbf{F}} \cdot d\vec{\mathbf{R}}}_{\text{Hard integral}} \overset{IoP}{=} \underbrace{\int_{C^*_{VL}} \vec{\mathbf{F}} \cdot d\vec{\mathbf{R}}^*_{VL} + \int_{C^*_{HL}} \vec{\mathbf{F}} \cdot d\vec{\mathbf{R}}^*_{HL}}_{\text{Easy integrals}}$$

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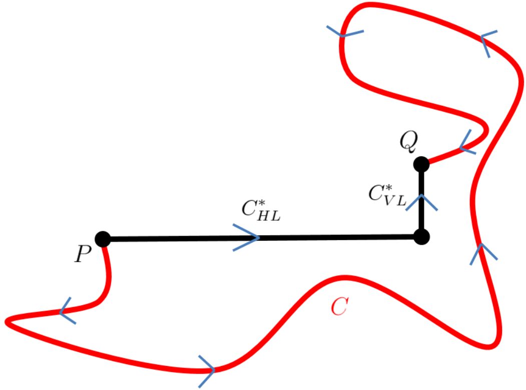

$$\underbrace{\int_C \vec{\mathbf{F}} \cdot d\vec{\mathbf{R}}}_{\text{Hard integral}} \overset{IoP}{=} \underbrace{\int_{C^*_{HL}} \vec{\mathbf{F}} \cdot d\vec{\mathbf{R}}^*_{HL} + \int_{C^*_{VL}} \vec{\mathbf{F}} \cdot d\vec{\mathbf{R}}^*_{VL}}_{\text{Easy integrals}}$$

Fin

Fin.