Green's Theorem

Calculus III

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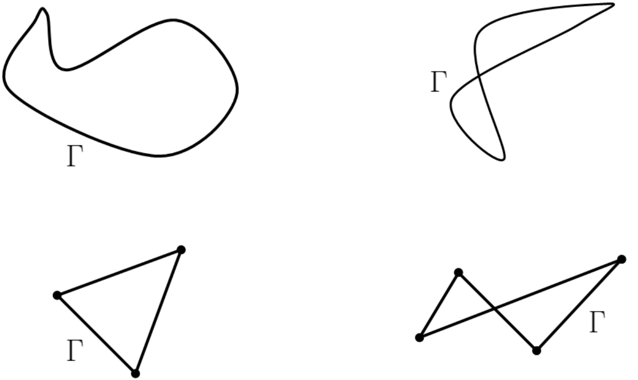
TTU

19 November 2014

Jordan Curves

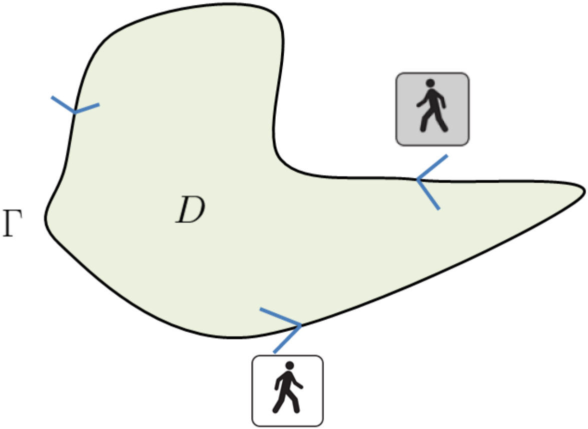


Jordan Curves | Not Jordan Curves

Definition

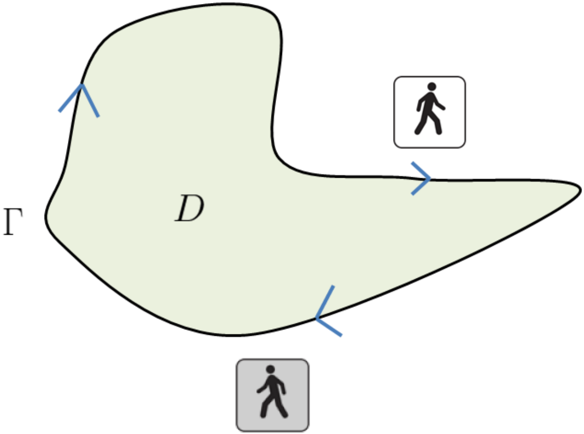
A **Jordan curve** is a piecewise smooth closed curve that does not intersect itself.

Positively Oriented Boundary Curves



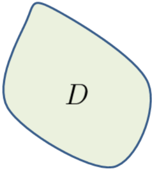
Positively Oriented Boundary Curve Γ
Region D is always on the left of the curve Γ

Negatively Oriented Boundary Curves

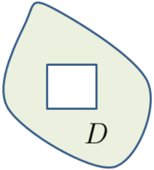


Negatively Oriented Boundary Curve Γ
Region D is always on the right of the curve Γ

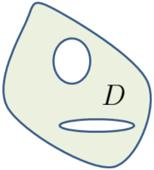
A Simply-Connected Region has "No Holes"



Simply-Connected Region

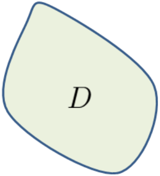


Doubly-Connected Region

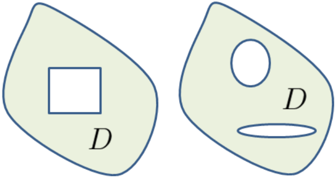


Triply-Connected Region

A Simply-Connected Region has "No Holes"

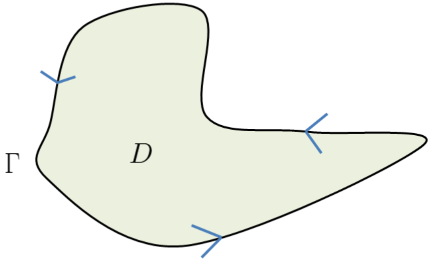


Simply-Connected Region



Multiply-Connected Regions

Green's Theorem (for Simply-Connected Regions)



Theorem

*Let $D \subset \mathbb{R}^2$ be a **simply-connected region** in the xy-plane.*
*Let Γ be a positively oriented **Jordan curve** that bounds region D.*
Let vector field $\vec{\mathbf{F}} \in C^{(1,1)}(D)$ s.t. $\vec{\mathbf{F}}(x,y) = \langle M(x,y), N(x,y) \rangle$.
Then

$$\oint_\Gamma (M\,dx + N\,dy) = \iint_D \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right)\,dA$$

Proof of Green's Theorem (via Bootstrapping)

Let's prove Green's Theorem for a **simply-connected** region D.

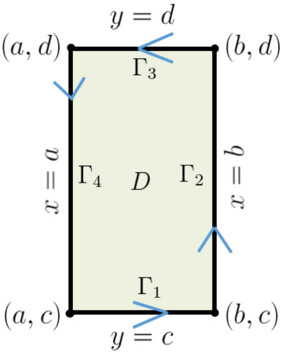
The proof uses the technique of **bootstrapping**, meaning several simpler cases are proven, and then more general (complicated) cases are proven by expressing them in terms of the simpler cases.

Here's how bootstrapping will proceed with proving Green's Theorem:

1. Prove Green's Theorem for **rectangular** regions.
2. Prove Green's Theorem for regions that are **both V-Simple & H-Simple**.
3. Prove Green's Theorem for any **simply-connected** region.

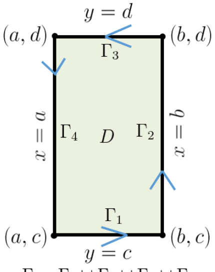
Proof of Green's Theorem (via Bootstrapping)

PART I: Suppose enclosed region D is a **rectangle**.



$$\Gamma = \Gamma_1 \cup \Gamma_2 \cup \Gamma_3 \cup \Gamma_4$$

PART I: Suppose enclosed region D is a **rectangle**.

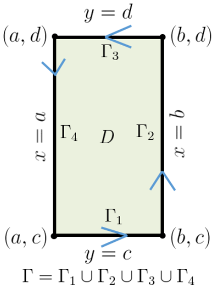


Parameterize subpaths $\Gamma_1, \ldots, \Gamma_4$:

$$\begin{aligned}
\Gamma_1 &: \vec{\mathbf{R}}_1(t) = \langle t, c\rangle \text{ for } t \in [a,b] \implies d\vec{\mathbf{R}}_1(t) = \langle 1, 0\rangle\, dt \implies dy = 0\\
\Gamma_2 &: \vec{\mathbf{R}}_2(t) = \langle b, t\rangle \text{ for } t \in [c,d] \implies d\vec{\mathbf{R}}_2(t) = \langle 0, 1\rangle\, dt \implies dx = 0\\
-\Gamma_3 &: \vec{\mathbf{R}}_3(t) = \langle t, d\rangle \text{ for } t \in [a,b] \implies d\vec{\mathbf{R}}_3(t) = \langle 1, 0\rangle\, dt \implies dy = 0\\
-\Gamma_4 &: \vec{\mathbf{R}}_4(t) = \langle a, t\rangle \text{ for } t \in [c,d] \implies d\vec{\mathbf{R}}_4(t) = \langle 0, 1\rangle\, dt \implies dx = 0
\end{aligned}$$

Proof of Green's Theorem (via Bootstrapping)

PART I: Suppose enclosed region D is a **rectangle**.



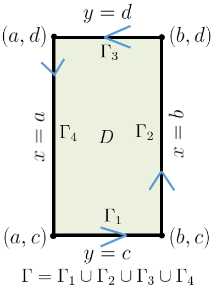
$$I_1 = \iint_D \frac{\partial M}{\partial y}\, dA = \int_a^b \int_c^d \frac{\partial M}{\partial y} dy\, dx = \int_a^b \Big[M(x,y) \Big]_{y=c}^{y=d} dx$$

$$\overset{FTC}{=} \int_a^b [M(x,d) - M(x,c)]\, dx = \int_a^b M(t,d)\, dt - \int_a^b M(t,c)\, dt$$

$$= \int_{-\Gamma_3} M\, dx - \int_{\Gamma_1} M\, dx = -\int_{\Gamma_3} M\, dx - \int_{\Gamma_1} M\, dx = -\oint_{\Gamma} M\, dx$$

Proof of Green's Theorem (via Bootstrapping)

PART I: Suppose enclosed region D is a **rectangle**.



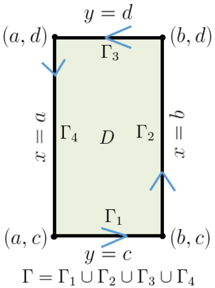
$$I_2 = \iint_D \frac{\partial N}{\partial x}\, dA = \int_c^d \int_a^b \frac{\partial N}{\partial x} dx\, dy = \int_c^d \Big[N(x,y) \Big]_{x=a}^{x=b} dy$$

$$\overset{FTC}{=} \int_c^d [N(b,y) - N(a,y)]\, dy = \int_c^d N(b,t)\, dt - \int_c^d N(a,t)\, dt$$

$$= \int_{\Gamma_2} N\, dy - \int_{-\Gamma_4} N\, dy = \int_{\Gamma_2} N\, dy + \int_{\Gamma_4} N\, dy = \oint_\Gamma N\, dy$$

Proof of Green's Theorem (via Bootstrapping)

PART I: Suppose enclosed region D is a **rectangle**.



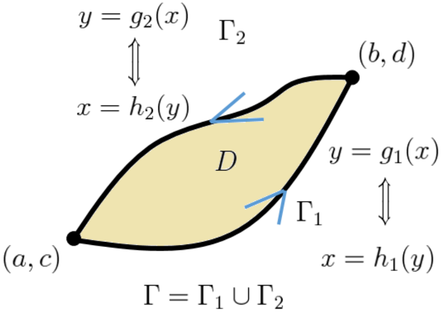
$$\therefore \iint_D \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dA = \iint_D \frac{\partial N}{\partial x} dA - \iint_D \frac{\partial M}{\partial y} dA$$

$$= \oint_\Gamma N\, dy - \left(- \oint_\Gamma M\, dx \right) = \oint_\Gamma (M\, dx + N\, dy)$$

QED

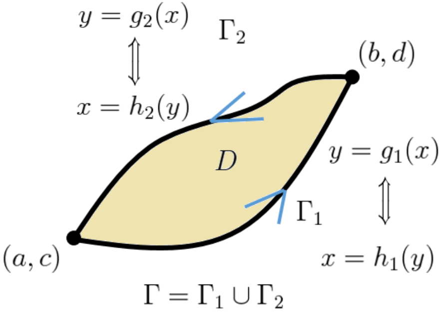
Proof of Green's Theorem (via Bootstrapping)

<u>PART II:</u> Suppose enclosed region D is **both V-Simple & H-Simple**.



Proof of Green's Theorem (via Bootstrapping)

PART II: Suppose enclosed region D is **both V-Simple & H-Simple**.



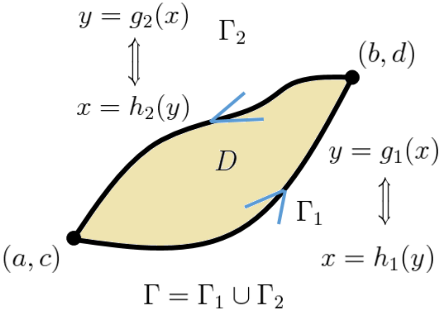
Parameterize subpaths Γ_1, Γ_2 two ways:

$$\Gamma_1 : \vec{\mathbf{R}}_1(t) = \langle t, g_1(t) \rangle \text{ for } t \in [a,b] \implies d\vec{\mathbf{R}}_1(t) = \langle 1, g_1'(t) \rangle \, dt \implies dx = dt$$
$$-\Gamma_2 : \vec{\mathbf{R}}_2(t) = \langle t, g_2(t) \rangle \text{ for } t \in [a,b] \implies d\vec{\mathbf{R}}_2(t) = \langle 1, g_2'(t) \rangle \, dt \implies dx = dt$$

$$\Gamma_1 : \vec{\mathbf{R}}_1(t) = \langle h_1(t), t \rangle \text{ for } t \in [c,d] \implies d\vec{\mathbf{R}}_1(t) = \langle h_1'(t), 1 \rangle \, dt \implies dy = dt$$
$$-\Gamma_2 : \vec{\mathbf{R}}_2(t) = \langle h_2(t), t \rangle \text{ for } t \in [c,d] \implies d\vec{\mathbf{R}}_2(t) = \langle h_2'(t), 1 \rangle \, dt \implies dy = dt$$

Proof of Green's Theorem (via Bootstrapping)

PART II: Suppose enclosed region D is **both V-Simple & H-Simple.**



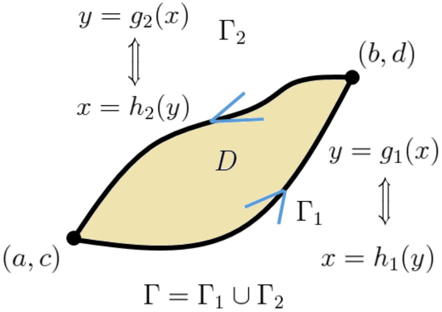
$$I_1 = \iint_D \frac{\partial M}{\partial y}\, dA = \int_a^b \int_{g_1(x)}^{g_2(x)} \frac{\partial M}{\partial y} dy\, dx = \int_a^b \Big[M(x,y)\Big]_{y=g_1(x)}^{y=g_2(x)} dx$$

$$\overset{FTC}{=} \int_a^b [M(x, g_2(x)) - M(x, g_1(x))]\, dx = \int_a^b M(t, g_2(t))\, dt - \int_a^b M(t, g_1(t))\, dt$$

$$= \int_{-\Gamma_2} M\, dx - \int_{\Gamma_1} M\, dx = -\int_{\Gamma_2} M\, dx - \int_{\Gamma_1} M\, dx = -\oint_{\Gamma} M\, dx$$

Proof of Green's Theorem (via Bootstrapping)

PART II: Suppose enclosed region D is **both V-Simple & H-Simple.**



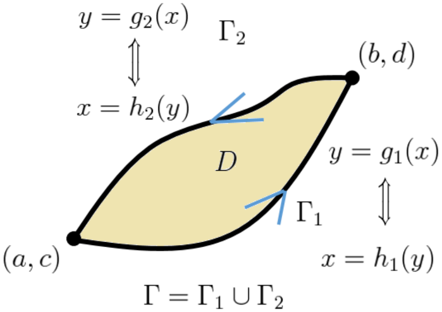
$$I_2 = \iint_D \frac{\partial N}{\partial x}\, dA = \int_c^d \int_{h_2(y)}^{h_1(y)} \frac{\partial N}{\partial x} dx\, dy = \int_c^d \Big[N(x,y) \Big]_{x=h_2(y)}^{x=h_1(y)} dy$$

$$\overset{FTC}{=} \int_c^d [N(h_1(y),y) - N(h_2(y),y)]\, dy = \int_c^d N(h_1(t),t)\, dt - \int_c^d N(h_2(t),t)\, dt$$

$$= \int_{\Gamma_1} N\, dy - \int_{-\Gamma_2} N\, dy = \int_{\Gamma_1} N\, dy + \int_{\Gamma_2} N\, dy = \oint_\Gamma N\, dy$$

Proof of Green's Theorem (via Bootstrapping)

PART II: Suppose enclosed region D is **both V-Simple & H-Simple.**

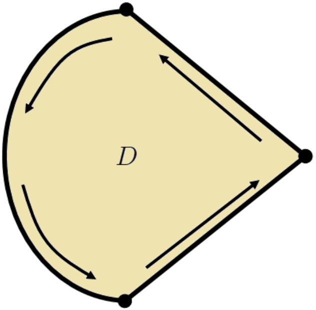


$$\therefore \iint_D \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) \, dA = \iint_D \frac{\partial N}{\partial x} dA - \iint_D \frac{\partial M}{\partial y} dA$$
$$= \oint_\Gamma N \, dy - \left(- \oint_\Gamma M \, dx \right) = \oint_\Gamma (M \, dx + N \, dy)$$

QED

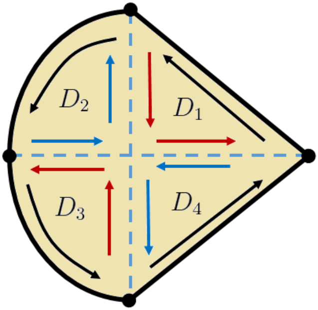
Proof of Green's Theorem (via Bootstrapping)

PART III: Suppose enclosed region D is **simply-connected**.



Proof of Green’s Theorem (via Bootstrapping)

<u>PART III:</u> Suppose enclosed region D is **simply-connected**.



$$D = D_1 \cup D_2 \cup D_3 \cup D_4$$

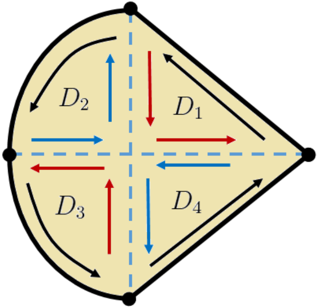
$$\Gamma = \Gamma_1 \cup \Gamma_2 \cup \Gamma_3 \cup \Gamma_4$$

where Γ_k is the positively-oriented path enclosing only subregion D_k.

Subdivide region D into rectangular and/or (V & H)-simple subregions.

Proof of Green's Theorem (via Bootstrapping)

PART III: Suppose enclosed region D is **simply-connected**.



$$D = D_1 \cup D_2 \cup D_3 \cup D_4$$

$$\Gamma = \Gamma_1 \cup \Gamma_2 \cup \Gamma_3 \cup \Gamma_4$$

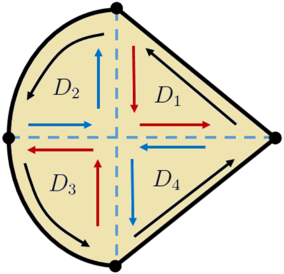
where Γ_k is the positively-oriented path enclosing only subregion D_k.

Observe that the line integral along a **blue arrow** exactly cancels with the same line integral along the **red arrow** pointing in the opposite direction.

What remains is the line integral along the **black arrows**: $\oint_\Gamma (M\,dx + N\,dy)$.

Proof of Green's Theorem (via Bootstrapping)

PART III: Suppose enclosed region D is **simply-connected**.



$$D = D_1 \cup D_2 \cup D_3 \cup D_4$$

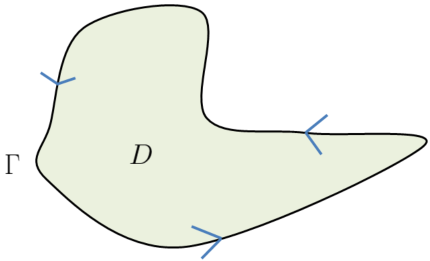
$$\Gamma = \Gamma_1 \cup \Gamma_2 \cup \Gamma_3 \cup \Gamma_4$$

where Γ_k is the positively-oriented path enclosing only subregion D_k.

$$\therefore \quad \oint_\Gamma (M\,dx + N\,dy) = \sum_{k=1}^{4} \oint_{\Gamma_k} (M\,dx + N\,dy)$$

$$= \sum_{k=1}^{4} \iint_{D_k} \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dA = \iint_D \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dA \qquad \square$$

Green's Theorem (for Simply-Connected Regions)



Theorem

Let $D \subset \mathbb{R}^2$ be a ***simply-connected region*** *in the xy-plane.*
Let Γ be a positively oriented ***Jordan curve*** *that bounds region D.*
Let vector field $\vec{\mathbf{F}} \in C^{(1,1)}(D)$ s.t. $\vec{\mathbf{F}}(x,y) = \langle M(x,y), N(x,y) \rangle$.
Then

$$\oint_\Gamma \vec{\mathbf{F}} \cdot d\vec{\mathbf{R}} = \iint_D (\nabla \times \vec{\mathbf{F}}) \cdot \widehat{\mathbf{k}} \, dA$$

Fin.