# Surface Integrals & Flux Integrals

## Calculus III

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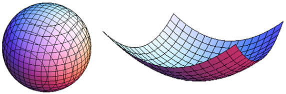
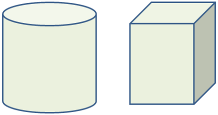
# PART I:

# SURFACE INTEGRALS

# Smooth Surfaces



Smooth Surfaces



Piecewise Smooth Surfaces

## Definition

A surface $S$ is **smooth** if the normal vector at each point exists & is nonzero.

## Definition

A surface $S$ is **piecewise smooth** if $S$ is the union of smooth subsurfaces.

# Surface Integrals

## Definition

Let **smooth** surface $S \subset \mathbb{R}^3$ be defined by $z = f(x, y)$.
Let region $D \subset \mathbb{R}^2$ be the projection of surface $S$ onto the $xy$-plane.
Let $f \in C^{(1,1)}(D)$ and $g \in C(S)$.

Then the **surface integral** of $g$ **across** $S$ is defined to be:

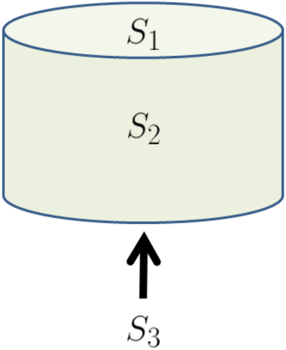
$$\underbrace{\iint_S g(x,y,z)\,dS}_{surface\ \ integral} := \underbrace{\iint_D g(x,y,f(x,y))\sqrt{1+(f_x)^2+(f_y)^2}\,dA}_{double\ \ integral}$$

REMARK:

$\iint_S dS$ gives the **surface area** of the portion of $S$ over region $D$ in $xy$-plane.

# Surface Integral Across a Piecewise Smooth Surface

$$S = S_1 \cup S_2 \cup S_3$$

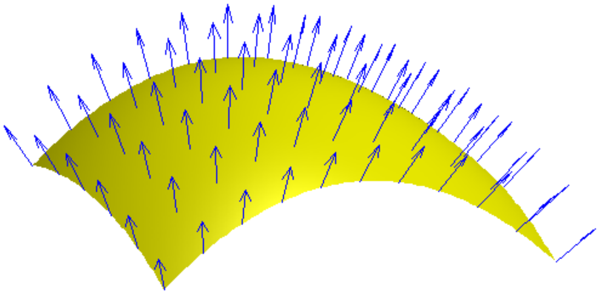



$$\iint_S g\,dS = \iint_{S_1} g\,dS + \iint_{S_2} g\,dS + \iint_{S_3} g\,dS$$

# PART II:

# FLUX INTEGRALS

# Orientable Surfaces



## Definition

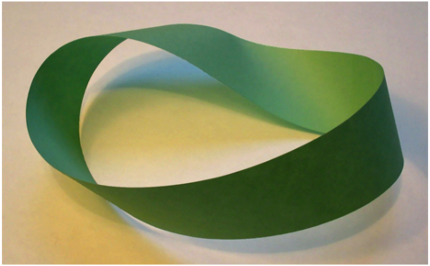
A surface $S$ is **orientable** if $S$ has a unit normal vector field $\widehat{\mathbf{N}}(x, y, z) \in C(S)$.

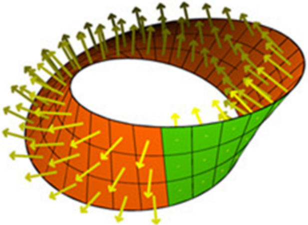
i.e. the unit normal vector field $\widehat{\mathbf{N}}$ must be continuous along the surface $S$.

Geometrically, this means as one traverses one side of the surface, the unit normal vector does not "flip direction."

REMARK: Most typical surfaces are orientable.

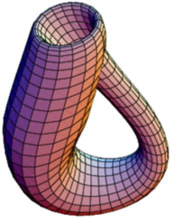
# Non-orientable Surfaces (Möbius Strip)

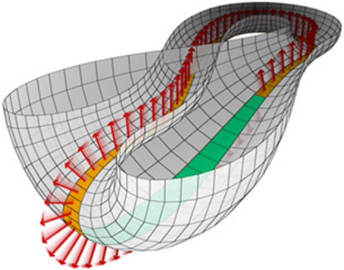




After one ”round trip” around the Möbius Strip, the unit normal vectors flip!

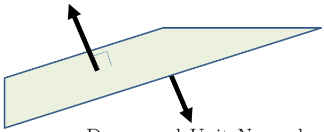
# Non-orientable Surfaces (Klein Bottle)





After one "round trip" around the Klein Bottle, the unit normal vector flips!

# Flux Integral of Surface $z = f(x, y)$




## Proposition

*Let orientable surface $S \subset \mathbb{R}^3$ be described by $z = f(x, y)$ w/ unit normal field $\widehat{\mathbf{N}}$.*
*Let region $D \subset \mathbb{R}^2$ be the projection of surface $S$ onto the $xy$-plane.*
*Let vector field $\vec{\mathbf{F}}(x, y, z) \in C^{(1,1,1)}(S)$ and scalar field $f \in C^{(1,1)}(D)$.*

*Then the **flux (integral) of $\vec{\mathbf{F}}$ across** $S$ is:*

$$\iint_S \vec{\mathbf{F}} \cdot \widehat{\mathbf{N}}\, dS = \iint_D \vec{\mathbf{F}}(x, y, f(x, y)) \cdot \langle -f_x, -f_y, 1 \rangle\, dA \qquad \text{(if } \widehat{\mathbf{N}} \text{ is upward)}$$

$$\iint_S \vec{\mathbf{F}} \cdot \widehat{\mathbf{N}}\, dS = \iint_D \vec{\mathbf{F}}(x, y, f(x, y)) \cdot \langle f_x, f_y, -1 \rangle\, dA \qquad \text{(if } \widehat{\mathbf{N}} \text{ is downward)}$$

# Flux Integral of Surface $z = f(x, y)$ (Proof)

## Proposition

*Let orientable surface $S \subset \mathbb{R}^3$ be described by $z = f(x, y)$ w/ unit normal field $\widehat{\mathbf{N}}$.*
*Let region $D \subset \mathbb{R}^2$ be the projection of surface $S$ onto the $xy$-plane.*
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$$\iint_S \vec{\mathbf{F}} \cdot \widehat{\mathbf{N}}\, dS = \iint_D \vec{\mathbf{F}}\,(x, y, f(x, y)) \cdot \langle f_x, f_y, -1\rangle\, dA \qquad \text{(if } \widehat{\mathbf{N}} \text{ is downward)}$$

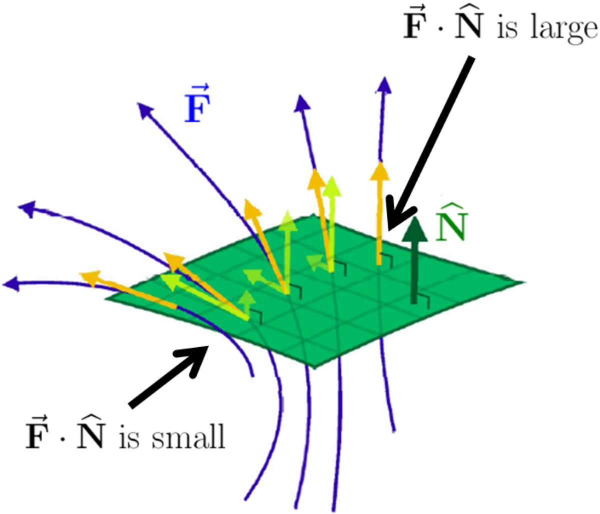
PROOF: Surface $z = f(x, y) \implies$ Level Surface $G(x, y, z) = z - f(x, y)$

$\implies$ Upward Unit Normal to Surface is $\widehat{\mathbf{N}} = \dfrac{\nabla G}{||\nabla G||} = \dfrac{\langle -f_x, -f_y, 1\rangle}{\sqrt{(-f_x)^2 + (-f_y)^2 + 1}}$

Recall that $dS = \sqrt{1 + (f_x)^2 + (f_y)^2}\, dA \implies \widehat{\mathbf{N}}\, dS = \langle -f_x, -f_y, 1\rangle\, dA$

$\therefore \iint_S \vec{\mathbf{F}} \cdot \widehat{\mathbf{N}}\, dS = \iint_D \vec{\mathbf{F}}(x, y, f(x, y)) \cdot \langle -f_x, -f_y, 1\rangle\, dA$ QED

# Flux (Interpretation)

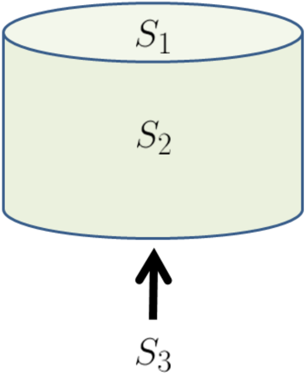



$\iint_S \vec{\mathbf{F}} \cdot \widehat{\mathbf{N}}\, dS$ is the total sum of all such dot products $\vec{\mathbf{F}} \cdot \widehat{\mathbf{N}}$ across $S$.

# Flux (Interpretation)

# Flux Integral across a Piecewise Smooth Surface

$$S = S_1 \cup S_2 \cup S_3$$




$$\iint_S \vec{\mathbf{F}} \cdot \widehat{\mathbf{N}}\, dS = \iint_{S_1} \vec{\mathbf{F}} \cdot \widehat{\mathbf{N}}\, dS + \iint_{S_2} \vec{\mathbf{F}} \cdot \widehat{\mathbf{N}}\, dS + \iint_{S_3} \vec{\mathbf{F}} \cdot \widehat{\mathbf{N}}\, dS$$

Fin.