Stokes' Theorem

Calculus III

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TTU

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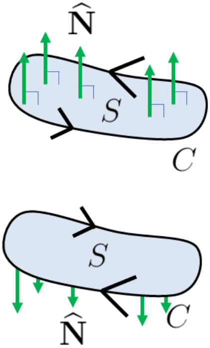
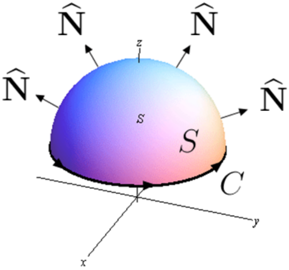
Compatible Orientations

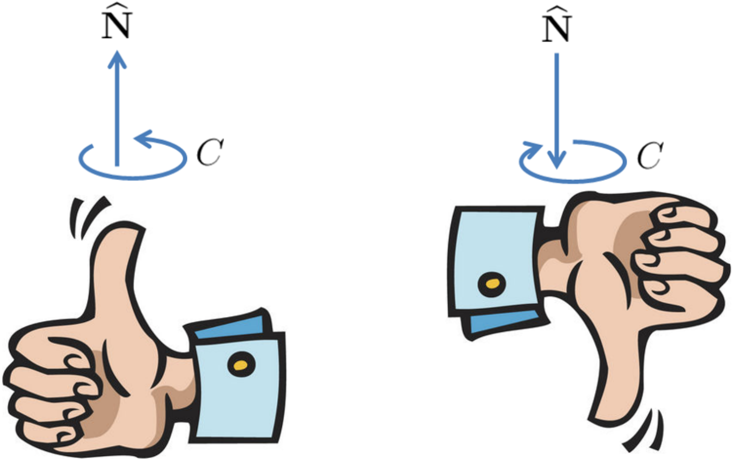




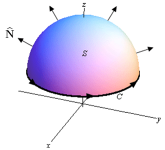
Definition

The orientations of a surface S & its bounding Jordan curve C are said to be **compatible** if the positive direction on C is counterclockwise in relation to the outward normal vector $\widehat{\mathbf{N}}$ of the surface

Compatible Orientations (Right-Hand Rule)



Stokes' Theorem



Theorem

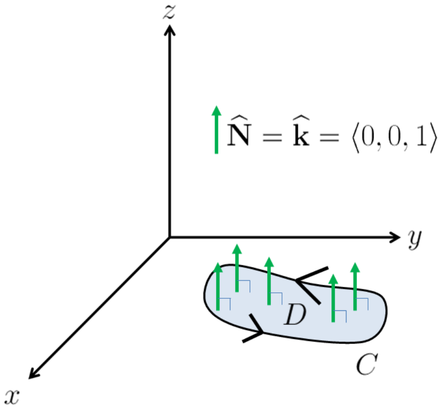
Let orientable surface $S \subset \mathbb{R}^3$ have the unit normal vector field $\widehat{\mathbf{N}}$.
Moreover, let S be bounded by a Jordan curve C whose orientation is compatible with the orientation on S.
Let vector field $\vec{\mathbf{F}} \in C^{(1,1,1)}(S)$.

Then:

$$\oint_C \vec{\mathbf{F}} \cdot d\vec{\mathbf{R}} = \iint_S \left(\nabla \times \vec{\mathbf{F}}\right) \cdot \widehat{\mathbf{N}}\, dS$$

<u>PROOF:</u> See the textbook if interested.

Stokes' Thm is a Generalization of Green's Theorem



Proposition

Green's Theorem is a special case of Stokes' Theorem.

Stokes' Thm is a Generalization of Green's Theorem

Proposition

Green's Theorem is a special case of Stokes' Theorem.

PROOF: Let vector field $\vec{\mathbf{F}}(x,y,z) = \langle M(x,y,z), N(x,y,z), P(x,y,z)\rangle$.

Green's Thm requires a region D on the xy-plane which is the surface $z = 0$.

Let $f(x,y) = 0 \implies dS = \sqrt{1 + (f_x)^2 + (f_y)^2}\, dA = \sqrt{1 + (0)^2 + (0)^2}\, dA = dA$.

WLOG assume bounding Jordan curve C to region D has positive orientation.
Then the unit normal field $\widehat{\mathbf{N}} = \widehat{\mathbf{k}} = \langle 0, 0, 1\rangle$.

$$\therefore \oint_C \vec{\mathbf{F}} \cdot d\vec{\mathbf{R}} \overset{STOKES}{=} \iint_D \left(\nabla \times \vec{\mathbf{F}}\right) \cdot \widehat{\mathbf{N}}\, dS$$

$$= \iint_D \left\langle \frac{\partial P}{\partial y} - \frac{\partial N}{\partial z}, \frac{\partial M}{\partial z} - \frac{\partial P}{\partial x}, \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right\rangle \cdot \langle 0, 0, 1\rangle\, dA$$

$$= \iint_D \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dA \qquad \text{QED}$$

WLOG ≡ "**W**ithout **L**oss **O**f **G**enerality"

The Value of Stokes' Theorem

Stokes' Thm converts a hard closed line integral to a simpler flux integral.

$$\underbrace{\oint_{\Gamma} \vec{\mathbf{F}} \cdot d\vec{\mathbf{R}}}_{\text{Hard closed line integral}} = \underbrace{\iint_{S} \left(\nabla \times \vec{\mathbf{F}}\right) \cdot \widehat{\mathbf{N}}\, dS}_{\text{Easier flux integral}}$$

But how to choose an appropriate surface S ??

BOUNDARY CURVE	**CANDIDATE SURFACE(S)**
Circle	Paraboloid, Half-Sphere, Circular Cone, or Plane
Ellipse	Paraboloid, Half-Ellipsoid, Elliptic Cone, or Plane
Polygon	Plane
Intersection of S_1 and S_2	Surface S_1 or S_2

The Value of Stokes' Theorem

Stokes' Thm converts a hard flux integral to a simpler closed line integral.

$$\underbrace{\iint_S \left(\nabla \times \vec{\mathbf{F}}\right) \cdot \widehat{\mathbf{N}}\, dS}_{\text{Hard flux integral}} \qquad = \qquad \underbrace{\oint_\Gamma \vec{\mathbf{F}} \cdot d\vec{\mathbf{R}}}_{\text{Easier closed line integral}}$$

Γ is the positively-oriented Jordan boundary curve of surface S.
Parameterize Γ as appropriate – typically, Γ is the intersection of S & a plane.

Fin

Fin.