Divergence Theorem (AKA Gauss' Theorem)

Calculus III

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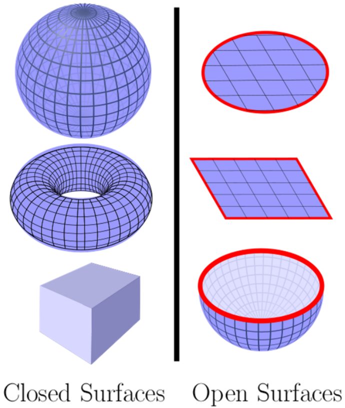
Open & Closed Surfaces in $\mathbb{R}^3$

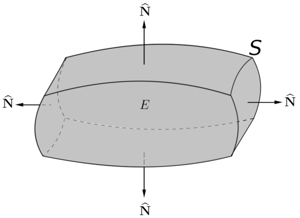


Definition

A **closed surface** $S \subset \mathbb{R}^3$ has no boundary curve.

An **open surface** $S \subset \mathbb{R}^3$ has a boundary curve (in red on above figure).

Divergence Theorem (AKA Gauss' Theorem)



Theorem

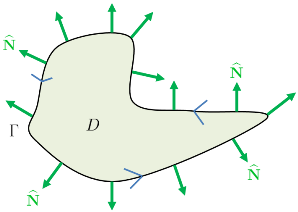
Let closed piecewise smooth surface S have outward unit normal $\widehat{\mathbf{N}}$.
Let simply-connected solid $E \subset \mathbb{R}^3$ be the interior of surface S.
i.e. S is the outer boundary surface of solid E.
Let vector field $\vec{\mathbf{F}} \in C^{(1,1,1)}(E)$.

Then:

$$\oiint_S \vec{\mathbf{F}} \cdot \widehat{\mathbf{N}}\, dS = \iiint_E \nabla \cdot \vec{\mathbf{F}}\, dV$$

PROOF: See the textbook if interested.

Gauss' Theorem on the xy-plane



Theorem

Let Jordan curve Γ be positively-oriented and have <u>*outward*</u> *unit normal $\widehat{\mathbf{N}}$.*
Let simply-connected region $D \subset \mathbb{R}^2$ be the interior of curve Γ.
i.e. Γ is the outer boundary curve of region D.
Let vector field $\vec{\mathbf{F}} \in C^{(1,1)}(D)$.

Then:

$$\oint_{\Gamma} \vec{\mathbf{F}} \cdot \widehat{\mathbf{N}}\, ds = \iint_{D} \nabla \cdot \vec{\mathbf{F}}\, dA$$

Gauss' Theorem on the xy-plane (Proof)

Theorem

Let Jordan curve Γ *be positively-oriented and have* <u>*outward*</u> *unit normal* $\widehat{\mathbf{N}}$*. Let simply-connected region* $D \subset \mathbb{R}^2$ *be the interior of* Γ*. Let* $\vec{\mathbf{F}} \in C^{(1,1)}(D)$*. Then:*

$$\oint_\Gamma \vec{\mathbf{F}} \cdot \widehat{\mathbf{N}}\, ds = \iint_D \nabla \cdot \vec{\mathbf{F}}\, dA$$

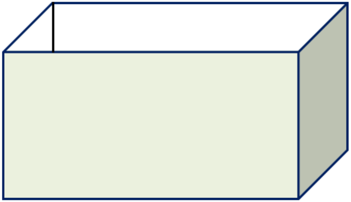
<u>PROOF:</u> Let $\vec{\mathbf{F}}(x,y) = \langle M(x,y), N(x,y)\rangle$ & $\Gamma:\ \vec{\mathbf{R}}(t) = \langle x(t), y(t)\rangle$, $t \in [a,b]$.

Then, $\vec{\mathbf{T}}(t) = \langle x'(t), y'(t)\rangle \implies \vec{\mathbf{T}}(t)\cdot\vec{\mathbf{N}}(t) = 0 \implies \vec{\mathbf{N}}(t) = \langle y'(t), -x'(t)\rangle$

$$\implies \widehat{\mathbf{N}}\, ds = \left(\frac{\langle y'(t), -x'(t)\rangle}{\sqrt{[y'(t)]^2 + [-x'(t)]^2}}\right)\left(\sqrt{[x'(t)]^2 + [y'(t)]^2}\, dt\right) = \langle y'(t), -x'(t)\rangle\, dt$$

$$\begin{aligned}
\therefore \oint_\Gamma \vec{\mathbf{F}}\cdot\widehat{\mathbf{N}}\, ds &= \int_a^b \langle M(x,y), N(x,y)\rangle \cdot \langle y'(t), -x'(t)\rangle\, dt \\
&= \oint_\Gamma \langle M, N\rangle \cdot \langle dy, -dx\rangle = \oint_\Gamma (-N)\, dx + M\, dy \\
&\overset{GREEN}{=} \iint_D \left[\frac{\partial M}{\partial x} - \left(-\frac{\partial N}{\partial y}\right)\right] dA = \iint_D \nabla\cdot\vec{\mathbf{F}}\, dA \qquad \text{QED}
\end{aligned}$$

Using Gauss' Theorem with Open Surfaces



Consider the above open box with the top face removed.

Without Gauss' Theorem:

$$\iint_{S_{Box}} \vec{\mathbf{F}} \cdot \widehat{\mathbf{N}}\, dS =$$

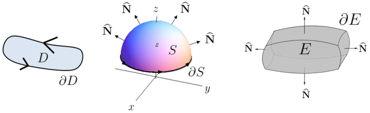
$$\iint_{S_{Bottom}} \vec{\mathbf{F}} \cdot \widehat{\mathbf{N}}\, dS + \iint_{S_{Left}} \vec{\mathbf{F}} \cdot \widehat{\mathbf{N}}\, dS + \iint_{S_{Right}} \vec{\mathbf{F}} \cdot \widehat{\mathbf{N}}\, dS + \iint_{S_{Front}} \vec{\mathbf{F}} \cdot \widehat{\mathbf{N}}\, dS + \iint_{S_{Back}} \vec{\mathbf{F}} \cdot \widehat{\mathbf{N}}\, dS$$

With Gauss' Theorem:

$$\iint_{S_{Box}} \vec{\mathbf{F}} \cdot \widehat{\mathbf{N}}\, dS = \iiint_{E_{Box}} \nabla \cdot \vec{\mathbf{F}}\, dV - \iint_{S_{Top}} \vec{\mathbf{F}} \cdot \widehat{\mathbf{N}}\, dS$$

Generalizations of the FTC

THEOREM	FTC FORM OF THEOREM	REMARK(S)
FTC	$\int_a^b f'(x)\,dx = f(b) - f(a)$	
FTLI	$\int_C \nabla f \cdot d\vec{\mathbf{R}} = f\left[\vec{\mathbf{R}}(b)\right] - f\left[\vec{\mathbf{R}}(a)\right]$	$C : \vec{\mathbf{R}}(t), a \leq t \leq b$
Green's Thm	$\iint_D \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dA = \oint_{\partial D} \vec{\mathbf{F}} \cdot d\vec{\mathbf{R}}$	$\vec{\mathbf{F}}(x,y) = \langle M(x,y), N(x,y)\rangle$
Stokes' Thm	$\iint_S \left(\nabla \times \vec{\mathbf{F}}\right) \cdot \widehat{\mathbf{N}}\, dS = \oint_{\partial S} \vec{\mathbf{F}} \cdot d\vec{\mathbf{R}}$	$S \equiv$ open surface
Gauss' Thm	$\iiint_E \nabla \cdot \vec{\mathbf{F}}\, dV = \oiint_{\partial E} \vec{\mathbf{F}} \cdot \widehat{\mathbf{N}}\, dS$	$E \equiv$ closed surface
Gauss' Thm	$\iint_D \nabla \cdot \vec{\mathbf{F}}\, dA = \oint_{\partial D} \vec{\mathbf{F}} \cdot \widehat{\mathbf{N}}\, ds$	$\vec{\mathbf{F}}(x,y) = \langle M(x,y), N(x,y)\rangle$



$\partial D \equiv$ Boundary curve of region D $\qquad$ $\partial S \equiv$ Boundary curve of surface S

$\partial E \equiv$ Boundary surface of solid E

Fin.