Vectors: Cross Products

Calculus III

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TTU

27 August 2014

2x2 & 3x3 Matrices (Determinant)

Definition

The **determinant** of a 2x2 matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is defined by:

$$\text{det}(A) = \begin{vmatrix} a & b \\ c & d \end{vmatrix} := ad - bc$$

Definition

The **determinant** of a 3x3 matrix $A = \begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix}$ is defined by:

$$\text{det}(A) = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} := a_1 \begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix} - a_2 \begin{vmatrix} b_1 & b_3 \\ c_1 & c_3 \end{vmatrix} + a_3 \begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix}$$

PROOF: Take Linear Algebra.

Cross Product (Definition)

Definition

The **cross product** of vectors $\vec{\mathbf{v}} = \langle v_1, v_2, v_3 \rangle$ and $\vec{\mathbf{w}} = \langle w_1, w_2, w_3 \rangle$ is:

$$\mathbf{v} \times \mathbf{w} := \begin{vmatrix} \widehat{\mathbf{i}} & \widehat{\mathbf{j}} & \widehat{\mathbf{k}} \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix} = \begin{vmatrix} v_2 & v_3 \\ w_2 & w_3 \end{vmatrix} \widehat{\mathbf{i}} - \begin{vmatrix} v_1 & v_3 \\ w_1 & w_3 \end{vmatrix} \widehat{\mathbf{j}} + \begin{vmatrix} v_1 & v_2 \\ w_1 & w_2 \end{vmatrix} \widehat{\mathbf{k}}$$

REMARKS:

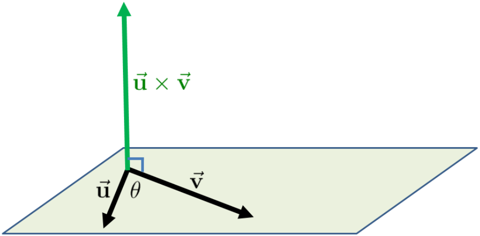
- The cross product $\mathbf{v} \times \mathbf{w}$ is a vector orthogonal to both vectors $\mathbf{v}$ and $\mathbf{w}$.
- **Cross products are defined only for 3D vectors!**

Cross Product (Properties)

Let vectors $\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbb{R}^3$ and scalars $s, t \in \mathbb{R}$. Then:

- $(s\mathbf{v}) \times (t\mathbf{w}) = st(\mathbf{v} \times \mathbf{w})$
- $\mathbf{v} \times \vec{\mathbf{0}} = \vec{\mathbf{0}} \times \mathbf{v} = \vec{\mathbf{0}}$
- $\mathbf{v} \times \mathbf{w} = -(\mathbf{w} \times \mathbf{v})$
- $\mathbf{v} \times \mathbf{v} = \vec{\mathbf{0}}$
- $\mathbf{u} \times (\mathbf{v} + \mathbf{w}) = (\mathbf{u} \times \mathbf{v}) + (\mathbf{u} \times \mathbf{w})$
- $(\mathbf{u} + \mathbf{v}) \times \mathbf{w} = (\mathbf{u} \times \mathbf{w}) + (\mathbf{v} \times \mathbf{w})$
- $||\mathbf{v} \times \mathbf{w}||^2 = ||\mathbf{v}||^2||\mathbf{w}||^2 - (\mathbf{v} \cdot \mathbf{w})^2$ (Lagrange's Identity)
- $\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = (\mathbf{c} \cdot \mathbf{a})\mathbf{b} - (\mathbf{b} \cdot \mathbf{a})\mathbf{c}$ ("cab-bac" Formula)

Cross Product (Geometric Interpretation)



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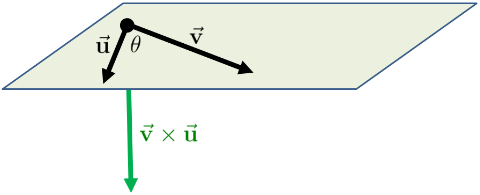
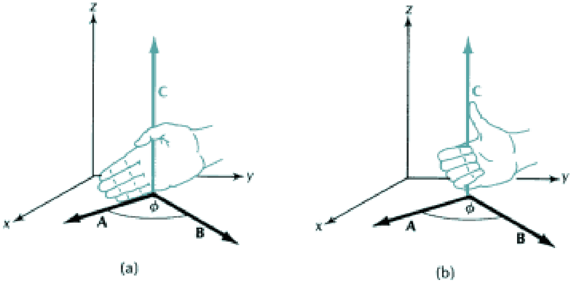


Cross Product (Right-Hand Rule)

$$\vec{\mathbf{C}} = \vec{\mathbf{A}} \times \vec{\mathbf{B}}$$



(a) Take right hand, stick thumb up & point fingers straight in direction of $\vec{\mathbf{A}}$.
(b) Curl fingers towards the direction of $\vec{\mathbf{B}}$, sweeping angle θ.

If performing part (b) is impossible, flip hand over and try again.
Thumb now points in the direction of the cross product $\vec{\mathbf{C}}$.

Cross Product (Coordinate-Free Definition)

Definition

Let non-zero vectors $\mathbf{v}, \mathbf{w} \in \mathbb{R}^3$, and $\theta \in [0, \pi]$ be the angle between them. Then:

$$\mathbf{v} \times \mathbf{w} = ||\mathbf{v}||||\mathbf{w}|| \sin(\theta)\widehat{\mathbf{n}}$$

$$||\mathbf{v} \times \mathbf{w}|| = ||\mathbf{v}||||\mathbf{w}|| \sin\theta$$

where **unit vector** $\widehat{\mathbf{n}}$ points in the direction of $\mathbf{v} \times \mathbf{w}$.

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where **unit vector** $\widehat{\mathbf{n}}$ points in the direction of $\mathbf{v} \times \mathbf{w}$.

PROOF:

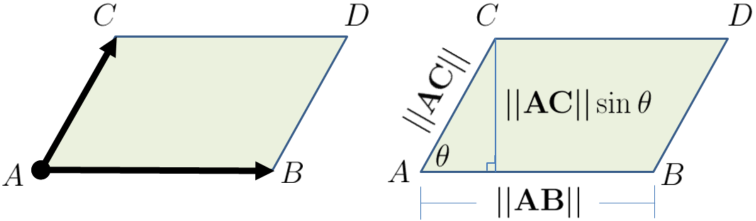
$$\begin{aligned}
||\mathbf{v} \times \mathbf{w}||^2 &= ||\mathbf{v}||^2||\mathbf{w}||^2 - (\mathbf{v} \cdot \mathbf{w})^2 && \text{(Lagrange's Identity)} \\
&= ||\mathbf{v}||^2||\mathbf{w}||^2 - (||\mathbf{v}||||\mathbf{w}|| \cos\theta)^2 && \text{(Coordinate-Free Dot Product)} \\
&= ||\mathbf{v}||^2||\mathbf{w}||^2 \left(1 - \cos^2\theta\right) && \text{(Square } 2^{nd} \text{ Term \& Factor RHS)} \\
&= ||\mathbf{v}||^2||\mathbf{w}||^2 \sin^2\theta && \text{(Trig Identity)}
\end{aligned}$$

$$\implies ||\mathbf{v} \times \mathbf{w}|| = ||\mathbf{v}||||\mathbf{w}|||\sin\theta|$$

$$\implies ||\mathbf{v} \times \mathbf{w}|| = ||\mathbf{v}||||\mathbf{w}|| \sin\theta \qquad \left(\text{Since } \theta \in [0, \pi] \implies \sin\theta \geq 0\right)$$

QED

Cross Product (Area of Parallelogram)



Parallelogram generated by nonzero nonparallel vectors $\mathbf{AB}$ & $\mathbf{AC}$

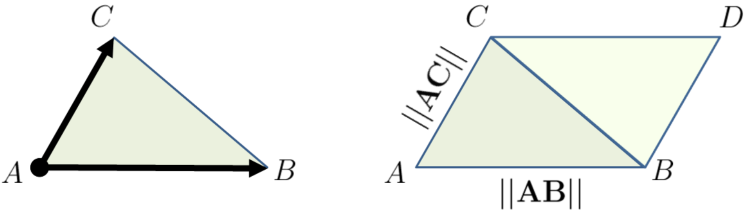
Area of Parallelogram = $\Big(\text{Base}\Big) \times \Big(\text{Height}\Big) = ||\mathbf{AB}||||\mathbf{AC}|| \sin\theta = ||\mathbf{AB} \times \mathbf{AC}||$

Theorem

$$\textit{Area of Parallelogram}(\mathbf{AB}, \mathbf{AC}) = ||\mathbf{AB} \times \mathbf{AC}||$$

- REMARK: Special Parallelograms $\rightarrow$ Squares, Rectangles, Rhombi

Cross Product (Area of Triangle)



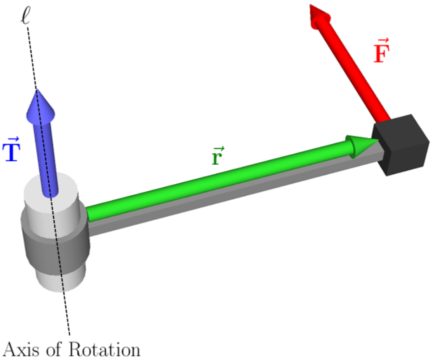
Triangle generated by nonzero nonparallel vectors **AB** & **AC**

Area of Triangle = $\frac{1}{2}\left(\text{Area of Parallelogram}\right) = \frac{1}{2}||\mathbf{AB} \times \mathbf{AC}||$

Theorem

$$\textit{Area of Triangle}(\mathbf{AB}, \mathbf{AC}) = \frac{1}{2}||\mathbf{AB} \times \mathbf{AC}||$$

Cross Product (Torque)



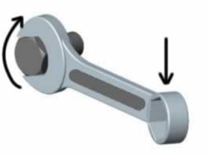
Definition

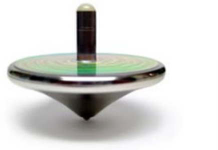
The **torque** $\vec{\mathbf{T}}$ of force $\vec{\mathbf{F}}$ applied a displacement $\vec{\mathbf{r}}$ from axis of rotation ℓ is:

$$\vec{\mathbf{T}} := \vec{\mathbf{r}} \times \vec{\mathbf{F}}$$

<u>PROOF:</u> Take Physics (Mechanics).

Cross Product (Torque)





Scalar Triple Product (Definition)

Definition

The **scalar triple product** of vectors $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbb{R}^3$ is defined by:

$$\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w}) := \begin{vmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix}$$

Scalar Triple Product (Definition)

Definition

The **scalar triple product** of vectors $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbb{R}^3$ is defined by:

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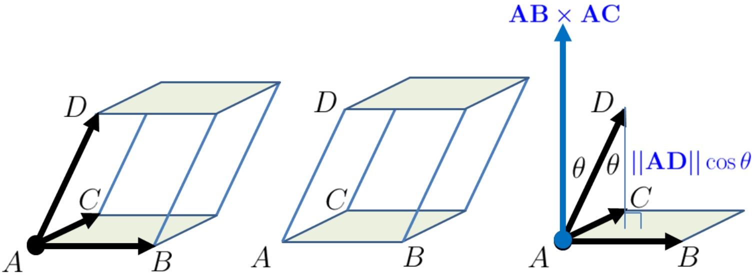
PROOF:

$$\begin{aligned} \mathbf{u} \cdot (\mathbf{v} \times \mathbf{w}) &= \left(u_1\widehat{\mathbf{i}} + u_2\widehat{\mathbf{j}} + u_3\widehat{\mathbf{k}}\right) \cdot \left(\begin{vmatrix} v_2 & v_3 \\ w_2 & w_3 \end{vmatrix} \widehat{\mathbf{i}} - \begin{vmatrix} v_1 & v_3 \\ w_1 & w_3 \end{vmatrix} \widehat{\mathbf{j}} + \begin{vmatrix} v_1 & v_2 \\ w_1 & w_2 \end{vmatrix} \widehat{\mathbf{k}} \right) \\ &= u_1 \begin{vmatrix} v_2 & v_3 \\ w_2 & w_3 \end{vmatrix} - u_2 \begin{vmatrix} v_1 & v_3 \\ w_1 & w_3 \end{vmatrix} + u_3 \begin{vmatrix} v_1 & v_2 \\ w_1 & w_2 \end{vmatrix} \\ &= \begin{vmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix} \end{aligned}$$

QED

Scalar Triple Product (Volume of Parallelopiped)



Parallelopiped generated by nonzero noncoplanar vectors $\mathbf{AB}, \mathbf{AC}, \mathbf{AD}$

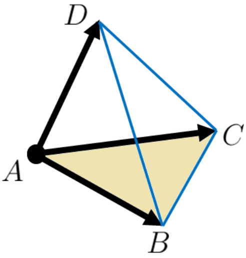
Volume of Parallelopiped =
$\Big(\text{Base Area}\Big) \times \Big(\text{Height}\Big) = ||\mathbf{AB} \times \mathbf{AC}||\,||\mathbf{AD}|| \cos\theta = |(\mathbf{AB} \times \mathbf{AC}) \cdot \mathbf{AD}|$

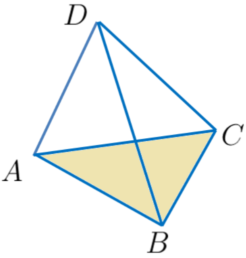
Theorem

Volume of Parallelopiped$(\mathbf{AB}, \mathbf{AC}, \mathbf{AD}) = |(\mathbf{AB} \times \mathbf{AC}) \cdot \mathbf{AD}|$

- REMARK: Special Parallelopipeds → Cubes, Rectangular Prisms

Scalar Triple Product (Volume of Tetrahedron)





$$\text{Volume of Tetrahedron} = \frac{1}{6}\Big(\text{Volume of Parallelopiped}\Big) = \frac{1}{6}|(\mathbf{AB} \times \mathbf{AC}) \cdot \mathbf{AD}|$$

Theorem

$$\textit{Volume of Tetrahedron}(\mathbf{AB}, \mathbf{AC}, \mathbf{AD}) = \frac{1}{6}|(\mathbf{AB} \times \mathbf{AC}) \cdot \mathbf{AD}|$$

Fin.