Parametric Curves, Lines in Space

Calculus III

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TTU

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PART I:
2D PARAMETRIC CURVES

Parametric Curves in $\mathbb{R}^2$ (Introduction)

Definition

A **2D parametric curve** has the following form:

$$\begin{cases} x = f(t) \\ y = g(t) \\ t \in I \end{cases}, \text{ where } \begin{array}{l} I \text{ is an } \textbf{interval} \\ f, g \in C(I) \end{array}$$

Each point (x, y) on the curve depends on a **parameter**, $t \in \mathbb{R}$.
A particular choice of functions f, g and interval I is called a **parameterization** of the curve.

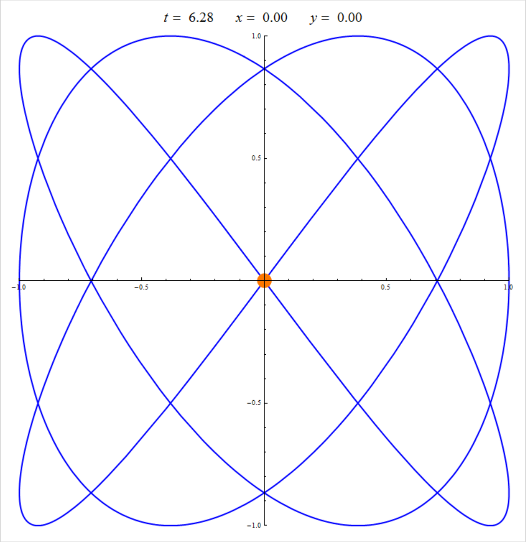
ALTERNATIVE NOTATION:

$$\begin{cases} x(t) = f(t) \\ y(t) = g(t) \\ t \in I \end{cases}$$ is used to emphasize that x and y are functions of t.

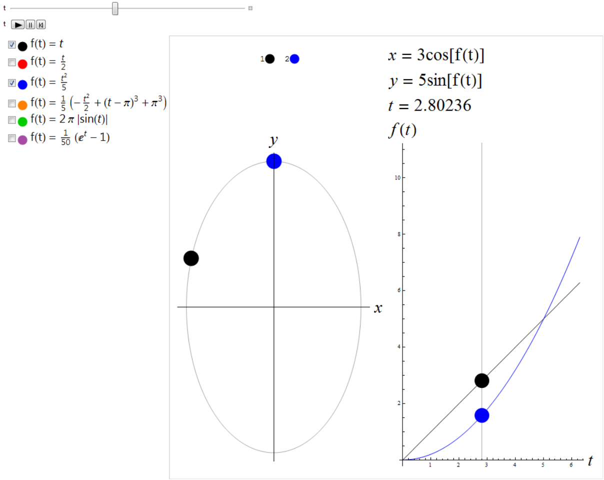
ALTERNATIVE NOTATION: $x = f(t), \ \ y = g(t), \ \ t \in I$

REMARK:
The "$t \in I$" piece is omitted when $t \in$ Dom$(f) \cap$ Dom(g)

(DEMO) PARAMETRIC CURVES (Click below):



(DEMO) COMPARING PARAMETERIZATIONS (Click below):



REMARK: Focus on choosing **simplest** parameterization (see next slide).

Parametric Curves in $\mathbb{R}^2$ (Conversions)

CONVERSION	PROCEDURE
2D Parametric $\to$ Rectangular	Solve for t in one eqn and substitute into the other eqn. Restrict the ranges of x & y if necessary.
Explicit Rectangular $\to$ 2D Parametric	$y = f(x) \implies \begin{cases} x = t \\ y = f(t) \\ t \in \text{Dom}(f) \end{cases}$
Explicit Rectangular $\to$ 2D Parametric	$x = g(y) \implies \begin{cases} x = g(t) \\ y = t \\ t \in \text{Dom}(g) \end{cases}$
Explicit Polar $\to$ 2D Parametric	$r = f(\theta) \implies \begin{cases} x = f(t)\cos t \\ y = f(t)\sin t \\ t \in \text{Dom}(f) \end{cases}$

***** ALL OTHER CONVERSION POSSIBILITIES ARE TOO DIFFICULT *****

Parametric Curves in $\mathbb{R}^3$ (Introduction)

Definition

A **3D parametric curve** has the following form:

$$\begin{cases} x = f(t) \\ y = g(t) \\ z = h(t) \\ t \in I \end{cases} \text{, where } \begin{array}{l} I \text{ is an } \textbf{interval} \\ f, g, h \in C(I) \end{array}$$

Each point (x, y, z) on the curve depends on a **parameter**, $t \in \mathbb{R}$.
A particular choice of functions f, g, h and interval I is called a **parameterization** of the curve.

REMARK:
General 3D parametric curves will be treated in Chapter 10.
For now, consider only **3D lines in space**.

PART II:
LINES IN SPACE

Lines in $\mathbb{R}^2$ (Canonical Forms from Algebra)

Recall from Algebra the following canonical forms of a line on the xy-plane:

TYPE	CANONICAL FORM	REMARK(s)
Point-Slope Form	$y - y_0 = m(x - x_0)$	$m \equiv$ Slope of Line Line contains point (x_0, y_0)
Slope-Intercept Form	$y = mx + b$	$m \equiv$ Slope of Line $b \equiv y$-Intercept : $(0, b)$
Both-Intercepts Form	$\frac{x}{a} + \frac{y}{b} = 1$	$m \equiv$ Slope of Line $a \equiv x$-Intercept : $(a, 0)$ $b \equiv y$-Intercept : $(0, b)$
General Form	$Ax + By = C$	$A, B, C \in \mathbb{R}$

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- Unfortunately, none of these forms are useful for **lines in space**.

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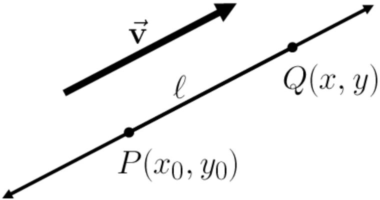
- Unfortunately, none of these forms are useful for **lines in space**.
- The solution: Use Vectors!

Lines (Parametric Form Derivation)

Given line $\ell \parallel$ vector $\vec{\mathbf{v}} = \langle v_1, v_2 \rangle$ and containing point $P(x_0, y_0)$



Let $Q(x, y)$ be any point on line ℓ.

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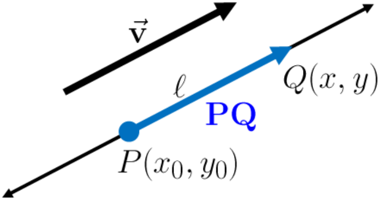

Form vector $\mathbf{PQ} = \langle x - x_0, y - y_0 \rangle$.

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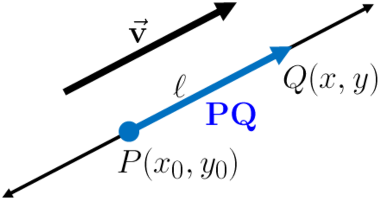

Notice that $\mathbf{PQ} \mid\mid \mathbf{v}$

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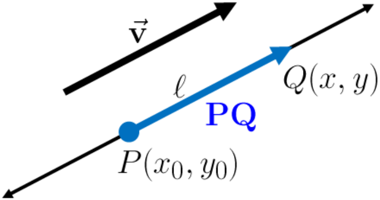

$$\mathbf{PQ} \mid\mid \mathbf{v} \implies \langle x - x_0, y - y_0 \rangle = t\langle v_1, v_2 \rangle$$

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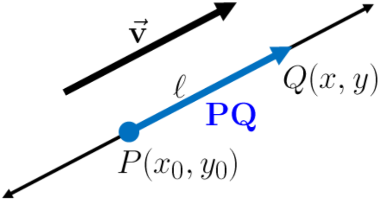

$$\mathbf{PQ} \parallel \mathbf{v} \implies \langle x - x_0, y - y_0 \rangle = t\langle v_1, v_2 \rangle \implies \langle x - x_0, y - y_0 \rangle = \langle tv_1, tv_2 \rangle$$

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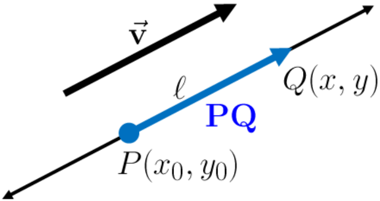

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Equate each component: $\begin{cases} x - x_0 = tv_1 \\ y - y_0 = tv_2 \end{cases}$

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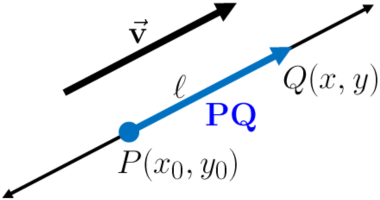
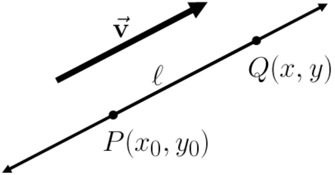


$$\mathbf{PQ} \parallel \mathbf{v} \implies \langle x - x_0, y - y_0 \rangle = t\langle v_1, v_2 \rangle \implies \langle x - x_0, y - y_0 \rangle = \langle tv_1, tv_2 \rangle$$

Equate each component: $\begin{cases} x - x_0 = tv_1 \\ y - y_0 = tv_2 \end{cases}$

Solve for (x, y): $\begin{cases} x = x_0 + tv_1 \\ y = y_0 + tv_2 \end{cases}$

Lines in $\mathbb{R}^2$ (Parametric Form)

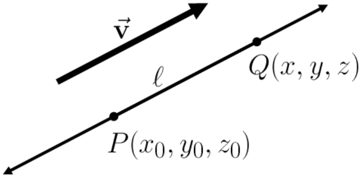


Definition

The **parametric form of line** ℓ containing point $P_0(x_0, y_0)$ and parallel to vector $\vec{\mathbf{v}} = \langle v_1, v_2 \rangle$ is:

$$\ell : \begin{cases} x = x_0 + v_1 t \\ y = y_0 + v_2 t \\ t \in \mathbb{R} \end{cases}$$

Lines in $\mathbb{R}^3$ (Parametric Form)



Definition

The **parametric form of line** ℓ containing point $P_0(x_0, y_0, z_0)$ and parallel to vector $\vec{\mathbf{v}} = \langle v_1, v_2, v_3 \rangle$ is:

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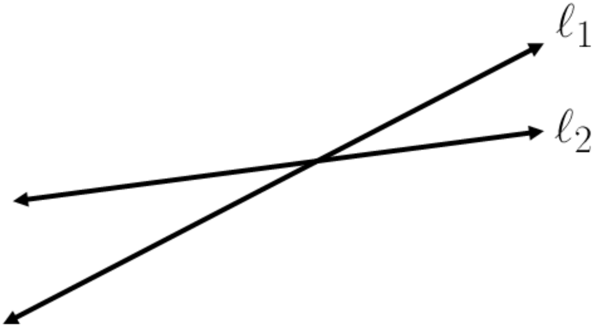
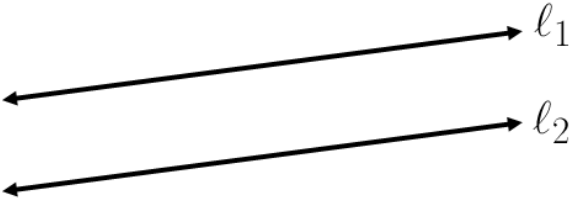
Lines in $\mathbb{R}^3$ (Intersecting Lines)

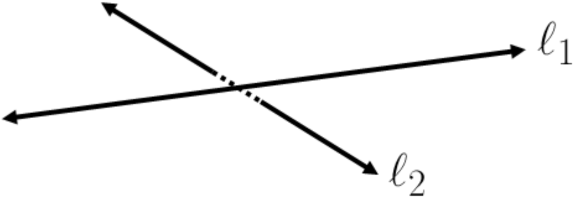


Lines in $\mathbb{R}^3$ (Parallel Lines)



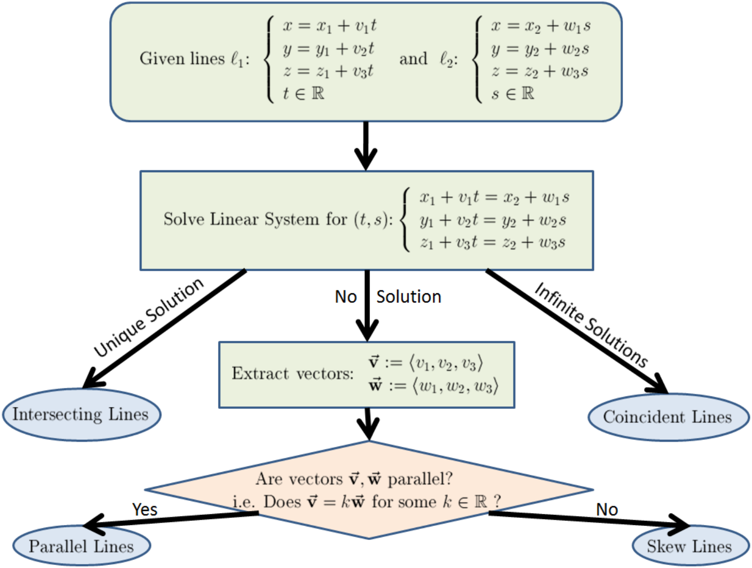
Lines in $\mathbb{R}^3$ (Coincident Lines)

Lines in $\mathbb{R}^3$ (Skew Lines)



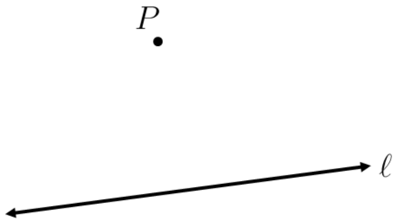
- The dashed line segment indicates that line ℓ_2 is behind line ℓ_1.

Lines in $\mathbb{R}^3$ (Classification Flowchart)



Lines in $\mathbb{R}^3$ (Distance from Point to Line Derivation)

Find the shortest distance from point P to line ℓ.



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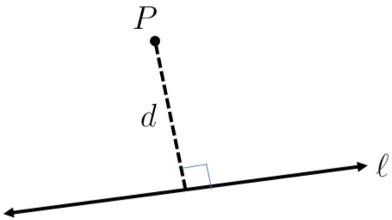

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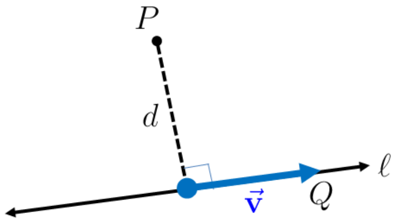

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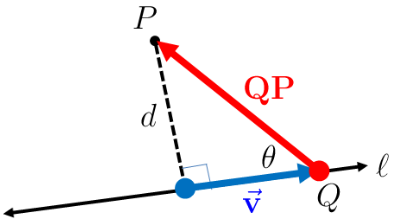

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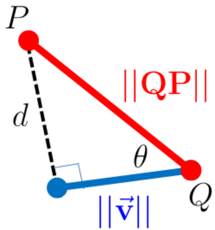

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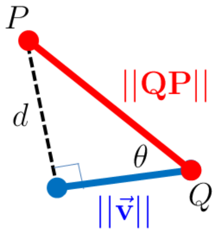

$$\sin\theta = \frac{d}{||\mathbf{QP}||}$$

Lines in $\mathbb{R}^3$ (Distance from Point to Line Derivation)

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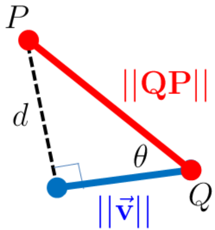

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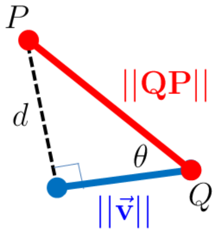

$$d||\mathbf{v}|| = ||\mathbf{v}||||\mathbf{QP}|| \sin\theta$$

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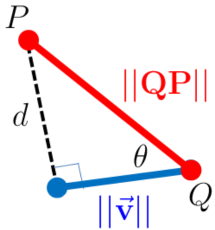

$$d||\mathbf{v}|| = ||\mathbf{v} \times \mathbf{QP}||$$

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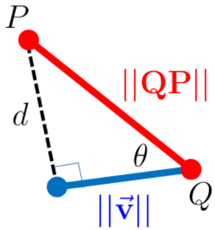
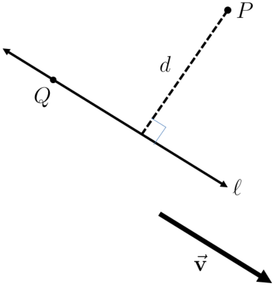
Find the shortest distance from point P to line ℓ.



$$d = \frac{||\mathbf{v} \times \mathbf{QP}||}{||\mathbf{v}||}$$

Lines in $\mathbb{R}^3$ (Distance from Point to Line)



Theorem

The shortest distance, d, from point P to line ℓ is:

$$d = \frac{||\mathbf{v} \times \mathbf{QP}||}{||\mathbf{v}||}$$

where Q is any point on line ℓ, and $\mathbf{v}$ is any vector parallel to line ℓ.

Fin

Fin.