Planes in Space

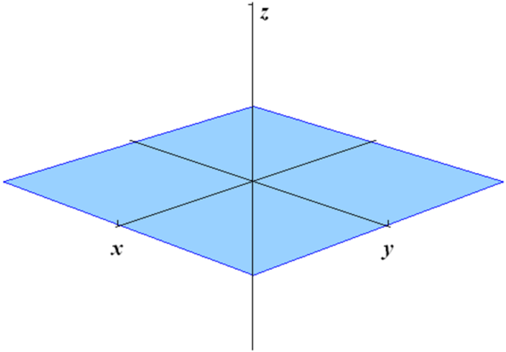
Calculus III

Josh Engwer

TTU

05 September 2014

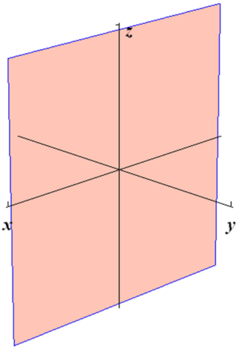
Special Planes (Coordinate Planes)



$$z = 0 \quad (xy\text{-plane})$$

- Mnemonic Device: $\Big(\text{missing variable}\Big) = 0$

Special Planes (Coordinate Planes)



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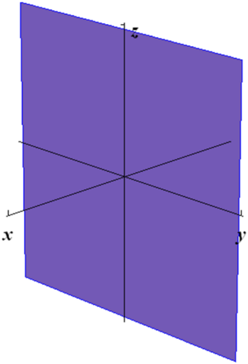
Special Planes (Coordinate Planes)



$x = 0 \quad (yz\text{-plane})$

- Mnemonic Device: $\Big(\text{missing variable}\Big) = 0$

Equations of Planes (Definition)

Definition

The **standard form** of a **plane** is:

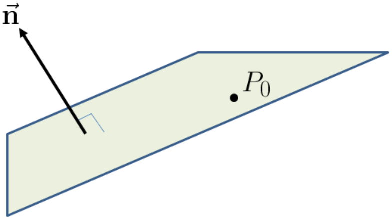
$$Ax + By + Cz + D = 0$$

where $A, B, C, D \in \mathbb{R}$.

Equations of Planes (CASE I)

Given plane with normal vector $\vec{\mathbf{n}} = \langle n_1, n_2, n_3 \rangle$ & containing point $P_0(x_0, y_0, z_0)$

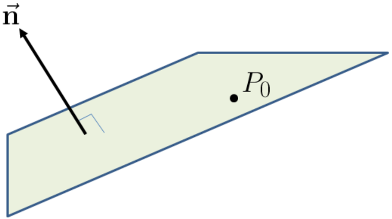
Equation of Plane: $Ax + By + Cz + D = 0$



Equations of Planes (CASE I)

Given plane with normal vector $\vec{\mathbf{n}} = \langle n_1, n_2, n_3 \rangle$ & containing point $P_0(x_0, y_0, z_0)$

Equation of Plane: $n_1 x + n_2 y + n_3 z + D = 0$

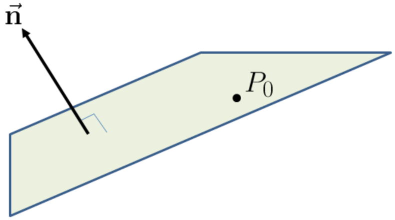


Let components of normal vector $\vec{\mathbf{n}}$ be the coefficients of x, y, z.

Equations of Planes (CASE I)

Given plane with normal vector $\vec{\mathbf{n}} = \langle n_1, n_2, n_3 \rangle$ & containing point $P_0(x_0, y_0, z_0)$

Equation of Plane: $n_1 x + n_2 y + n_3 z + D = 0$



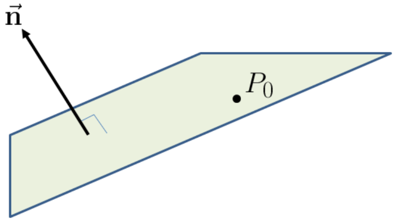
Substitute given point (x_0, y_0, z_0) into x, y, z respectively:

$$n_1 x_0 + n_2 y_0 + n_3 z_0 + D = 0$$

Equations of Planes (CASE I)

Given plane with normal vector $\vec{\mathbf{n}} = \langle n_1, n_2, n_3 \rangle$ & containing point $P_0(x_0, y_0, z_0)$

Equation of Plane: $n_1 x + n_2 y + n_3 z + (-n_1 x_0 - n_2 y_0 - n_3 z_0) = 0$

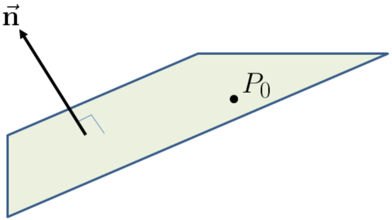


Solve the resulting equation for D: $D = -n_1 x_0 - n_2 y_0 - n_3 z_0$

Equations of Planes (CASE I)

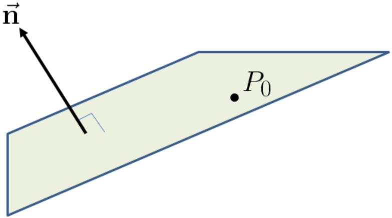
Given plane with normal vector $\vec{\mathbf{n}} = \langle n_1, n_2, n_3 \rangle$ & containing point $P_0(x_0, y_0, z_0)$

Equation of Plane: $n_1(x - x_0) + n_2(y - y_0) + n_3(z - z_0) = 0$



Group terms.

Equations of Planes (CASE I)



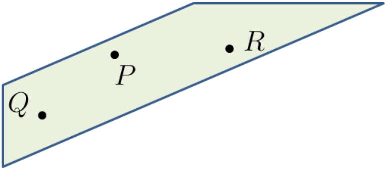
Proposition

Equation of plane with normal vector $\vec{\mathbf{n}} = \langle n_1, n_2, n_3 \rangle$ *and containing point* $P_0(x_0, y_0, z_0)$ *is:*

$$n_1(x - x_0) + n_2(y - y_0) + n_3(z - z_0) = 0$$

Equations of Planes (CASE II)

Given a plane containing three noncollinear points P, Q, R.



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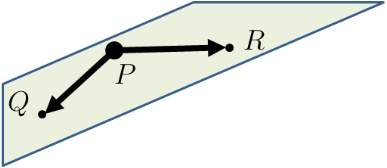
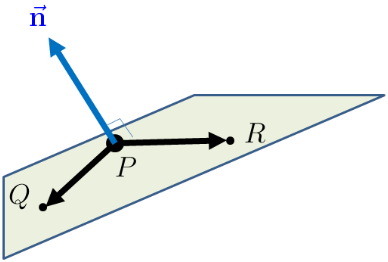


Form vectors $\mathbf{PQ}$ and $\mathbf{PR}$.

Equations of Planes (CASE II)

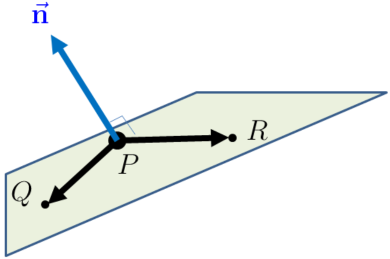
Given a plane containing three noncollinear points P, Q, R.



Form normal vector $\vec{\mathbf{n}}$ using a cross product: $\vec{\mathbf{n}} = \mathbf{PQ} \times \mathbf{PR}$

Equations of Planes (CASE II)

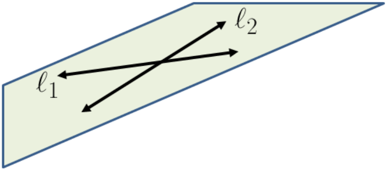
Given a plane containing three noncollinear points P, Q, R.



Using normal vector $\vec{\mathbf{n}}$ and one of the three given points P, Q, R, follow CASE I.

Equations of Planes (CASE III)

Given a plane containing intersecting lines ℓ_1, ℓ_2.



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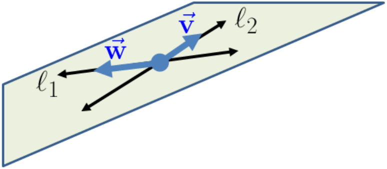
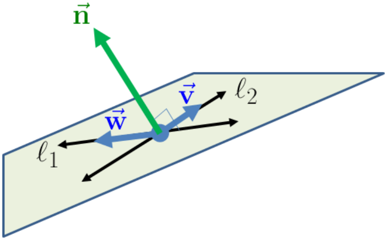


Form vectors $\vec{\mathbf{w}}$ and $\vec{\mathbf{v}}$ parallel to lines ℓ_1 and ℓ_2 respectively.

Equations of Planes (CASE III)

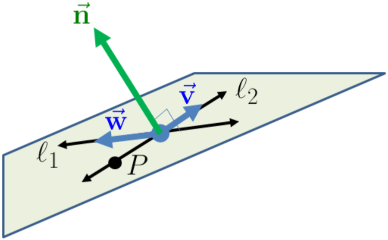
Given a plane containing intersecting lines ℓ_1, ℓ_2.



Form normal vector $\vec{\mathbf{n}}$ using a cross product: $\vec{\mathbf{n}} = \vec{\mathbf{v}} \times \vec{\mathbf{w}}$

Equations of Planes (CASE III)

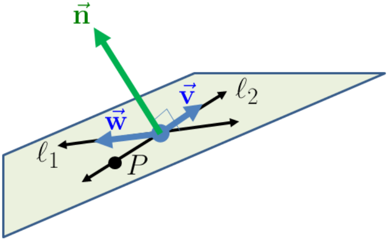
Given a plane containing intersecting lines ℓ_1, ℓ_2.



Pick any (simple) point P on either line ℓ_1 or ℓ_2.

Equations of Planes (CASE III)

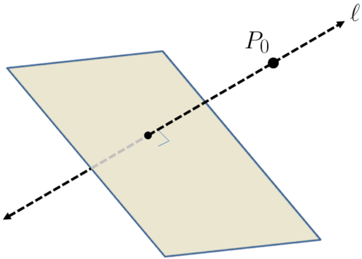
Given a plane containing intersecting lines ℓ_1, ℓ_2.



Using normal vector $\vec{\mathbf{n}}$ and point P, follow CASE I.

Line Orthogonal to a Plane (PART 1)

Given point $P_0(x_0, y_0, z_0)$ and plane $Ax + By + Cz + D = 0$



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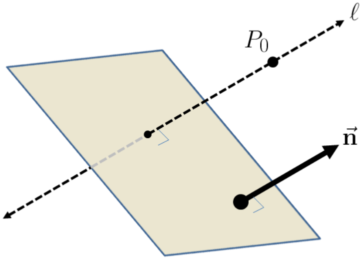
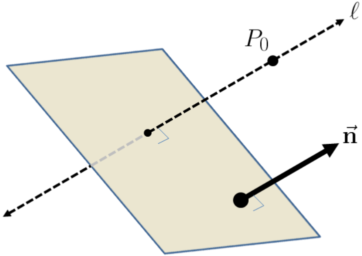


Extract normal vector from plane: $\vec{\mathbf{n}} = \langle A, B, C \rangle$.

Line Orthogonal to a Plane (PART 1)

Given point $P_0(x_0, y_0, z_0)$ and plane $Ax + By + Cz + D = 0$
Find an equation for the line ℓ.



Notice that line ℓ is parallel to normal vector $\vec{\mathbf{n}}$.

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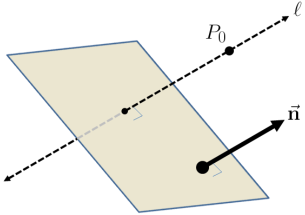
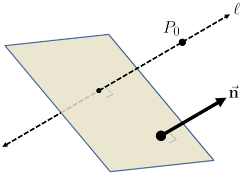


Equation of line ℓ containing point P_0 and orthogonal to plane is:

$$\begin{cases} x = x_0 + At \\ y = y_0 + Bt \\ z = z_0 + Ct \\ t \in \mathbb{R} \end{cases}$$

Line Orthogonal to a Plane (PART 1)



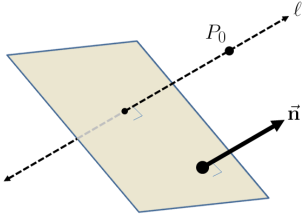
Proposition

Equation of line ℓ *containing point* $P_0(x_0, y_0, z_0)$ *and orthogonal to plane* $Ax + By + Cz + D = 0$ *is:*

$$\begin{cases} x = x_0 + At \\ y = y_0 + Bt \\ z = z_0 + Ct \\ t \in \mathbb{R} \end{cases}$$

Line Orthogonal to a Plane (PART 2)

Given point $P_0(x_0, y_0, z_0)$ and plane $Ax + By + Cz + D = 0$
Find the point of intersection of the line & plane.



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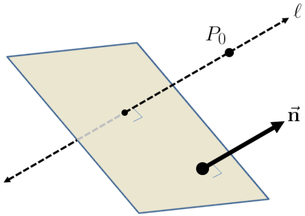

Substitute equation of line ℓ into equation of plane, then solve for parameter t.

Line Orthogonal to a Plane (PART 2)

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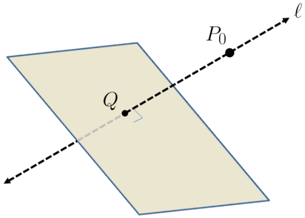
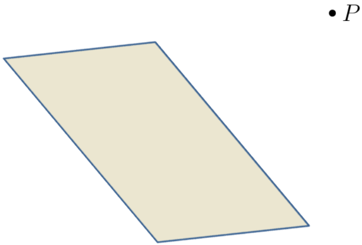


Plug value of t into equation of line ℓ to determine point Q where the line intersects plane.

Distance between Point and Plane (Derivation)

Given point P and plane $Ax + By + Cz + D = 0$



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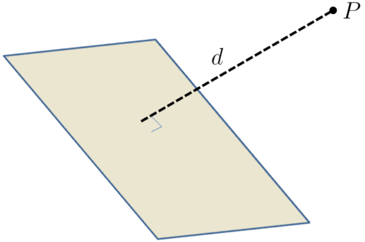

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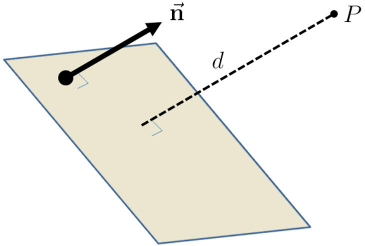

Extract normal vector from the plane: $\vec{\mathbf{n}} = \langle A, B, C \rangle$

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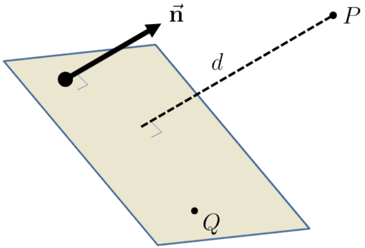
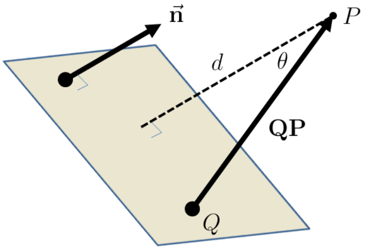


Pick any (simple) point Q on the plane.

Distance between Point and Plane (Derivation)

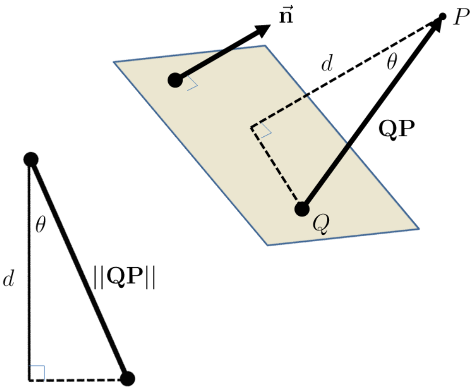
Given point P and plane $Ax + By + Cz + D = 0$



Form vector $\mathbf{QP}$.

Distance between Point and Plane (Derivation)

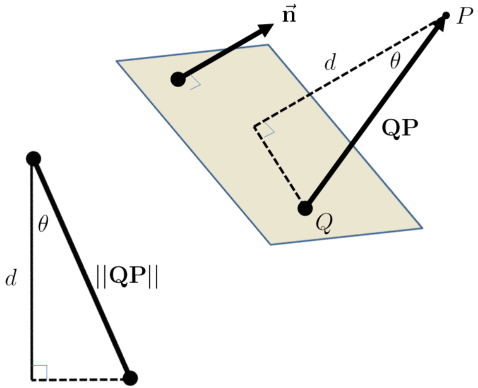
Given point P and plane $Ax + By + Cz + D = 0$



Form right triangle.

Distance between Point and Plane (Derivation)

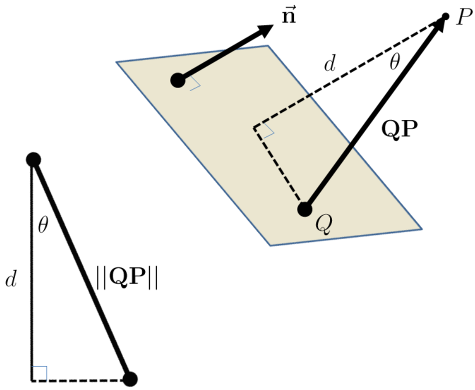
Given point P and plane $Ax + By + Cz + D = 0$



$$\cos\theta = \frac{d}{||\mathbf{QP}||}$$

Distance between Point and Plane (Derivation)

Given point P and plane $Ax + By + Cz + D = 0$



$$d = ||\mathbf{QP}|| \cos\theta$$

Distance between Point and Plane (Derivation)

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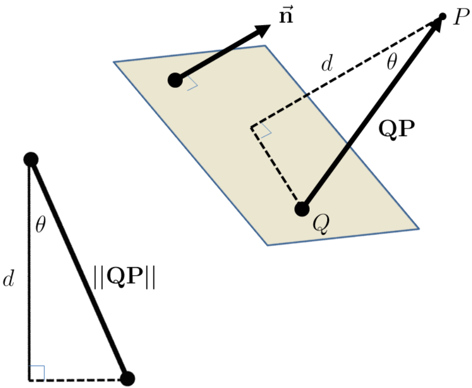
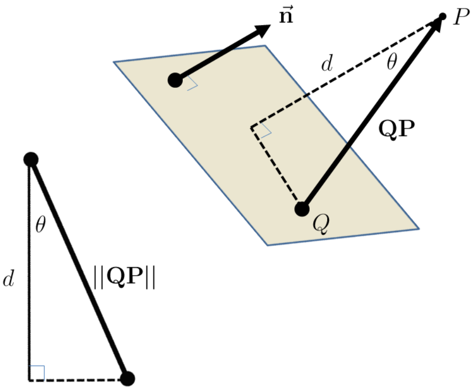


$$||\vec{\mathbf{n}}||d = ||\mathbf{QP}||||\vec{\mathbf{n}}|| \cos \theta$$

Distance between Point and Plane (Derivation)

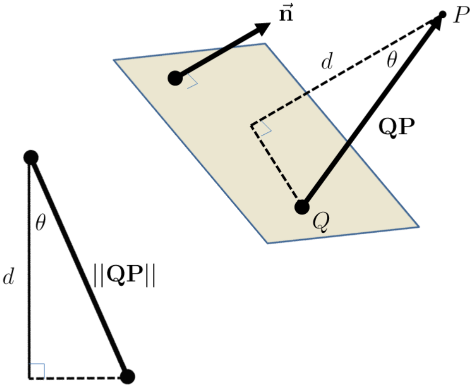
Given point P and plane $Ax + By + Cz + D = 0$



$$||\vec{\mathbf{n}}||d = |\mathbf{QP} \cdot \vec{\mathbf{n}}|$$

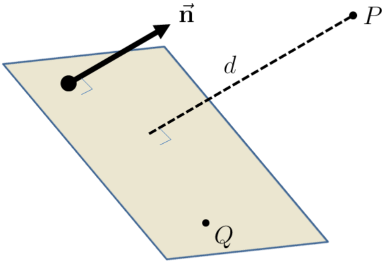
Distance between Point and Plane (Derivation)

Given point P and plane $Ax + By + Cz + D = 0$



$$d = \frac{|\mathbf{QP} \cdot \vec{\mathbf{n}}|}{||\vec{\mathbf{n}}||}$$

Distance between Point and Plane (Formula)



Proposition

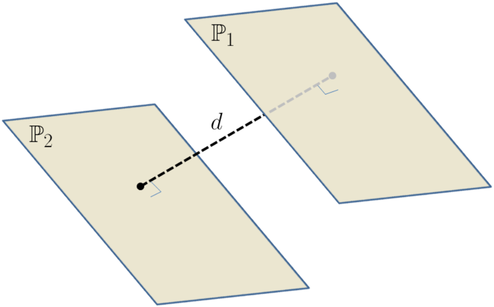
The distance, d, between a point P and a plane is:

$$d = \frac{|\mathbf{QP} \cdot \vec{\mathbf{n}}|}{||\vec{\mathbf{n}}||}$$

where $\vec{\mathbf{n}}$ is a ***normal vector*** *to the plane*
and Q is any (simple) point on the plane.

Distance between Two Parallel Planes

Given parallel planes
$\mathbb{P}_1 : A_1x + B_1y + C_1z + D_1 = 0$ and $\mathbb{P}_2 : A_2x + B_2y + C_2z + D_2 = 0$



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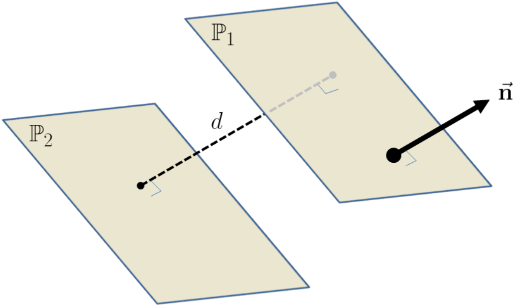
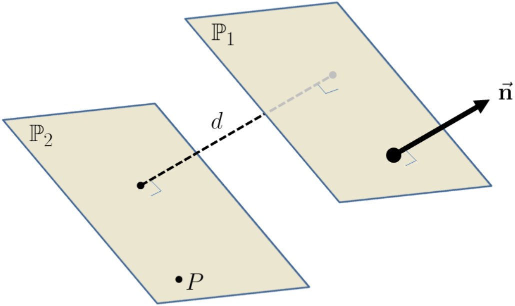


Extract normal vector to plane $\mathbb{P}_1$: $\vec{\mathbf{n}} = \langle A_1, B_1, C_1 \rangle$

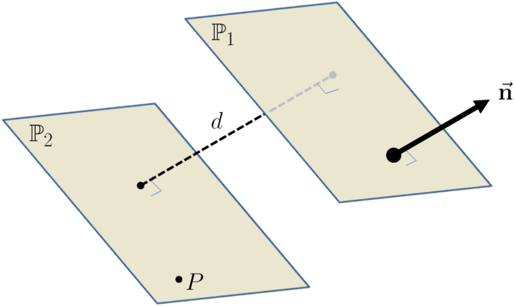
Given parallel planes
$\mathbb{P}_1 : A_1x + B_1y + C_1z + D_1 = 0$ and $\mathbb{P}_2 : A_2x + B_2y + C_2z + D_2 = 0$



Pick any (simple) point P on plane $\mathbb{P}_2$.

Distance between Two Parallel Planes

Given parallel planes
$\mathbb{P}_1 : A_1x + B_1y + C_1z + D_1 = 0$ and $\mathbb{P}_2 : A_2x + B_2y + C_2z + D_2 = 0$



Find the distance between point P and plane $\mathbb{P}_1$.

Fin.