Quadric Surfaces

Calculus III

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TTU

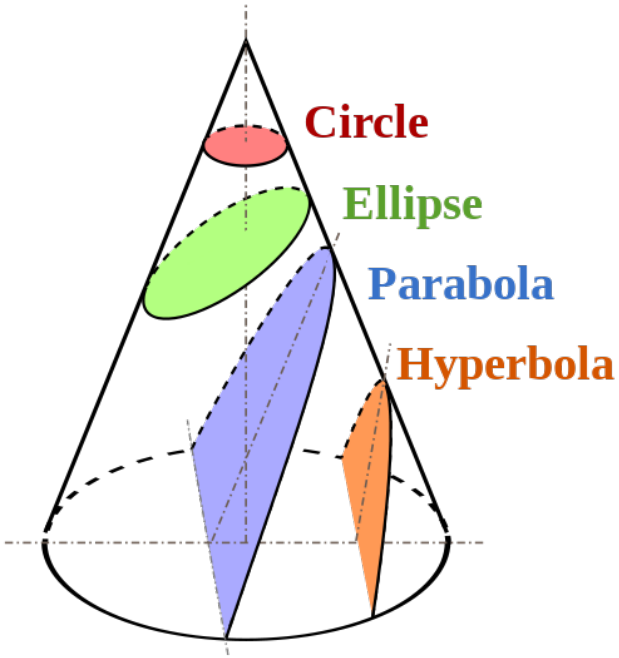
08 September 2014

PART I:

QUADRIC SURFACES
(IDENTIFICATION & CHARACTERIZATION)

Conic Sections (Definition)

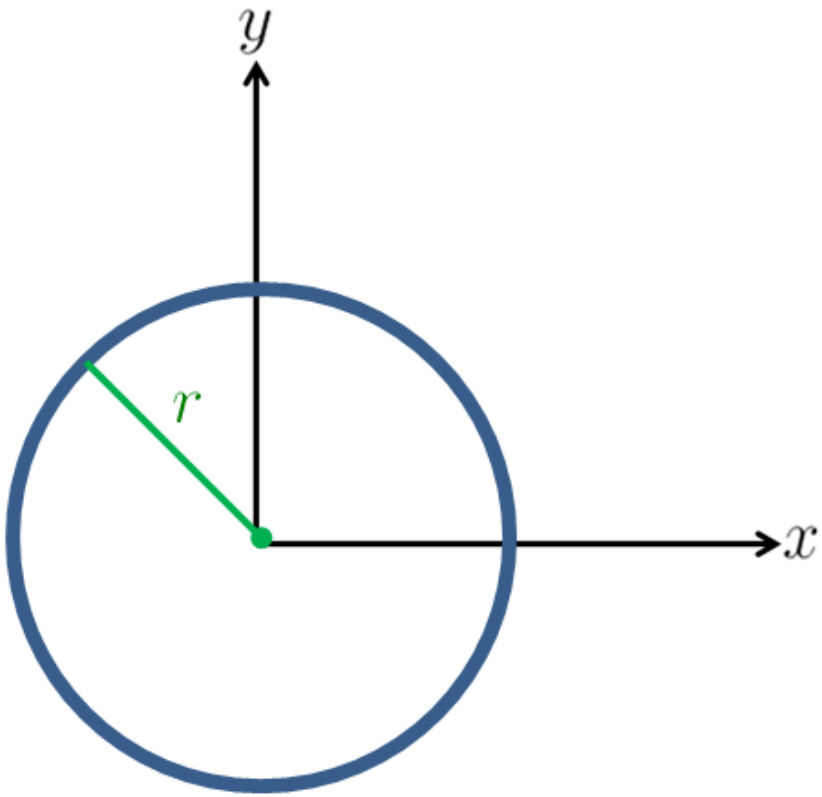
Recall from Algebra or PreCalculus the **conic sections**:



- General Form: $Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$
- The general form is much too general! So, recall the canonical forms.

Conic Sections (Circle)

Recall from Algebra or PreCalculus the **canonical form** of a **circle**:

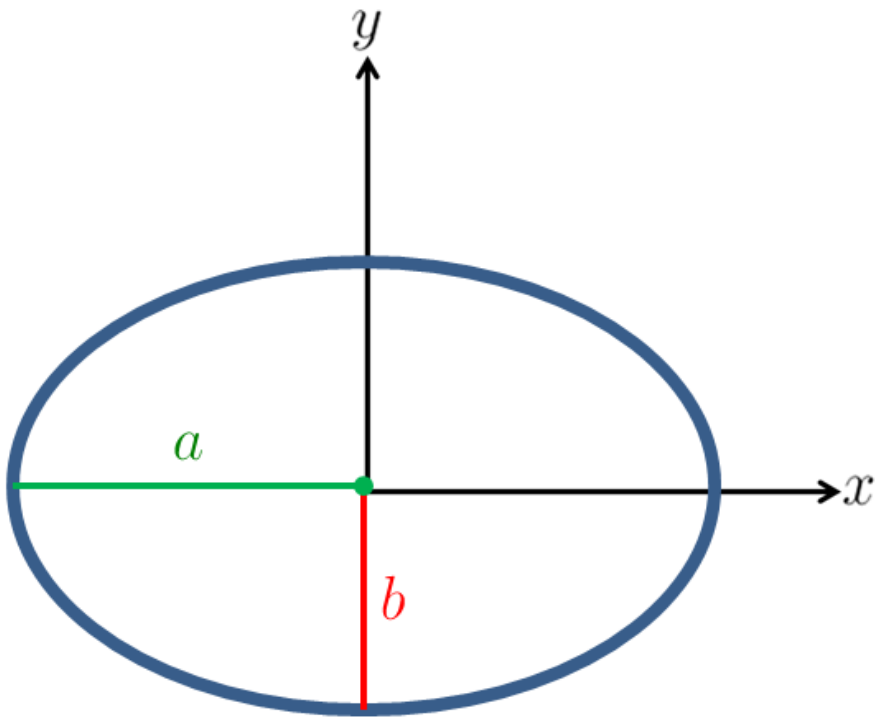


$$x^2 + y^2 = r^2$$

Radius: r

Conic Sections (Ellipse)

Recall from Algebra or PreCalculus the **canonical form** of an **ellipse**:

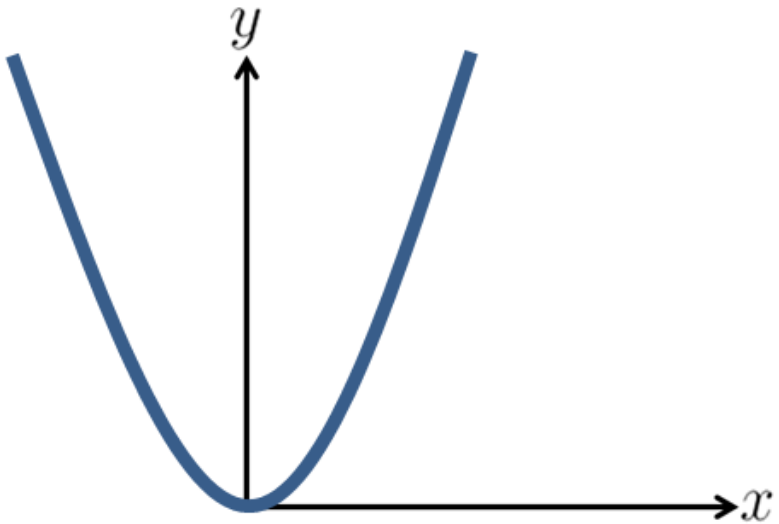


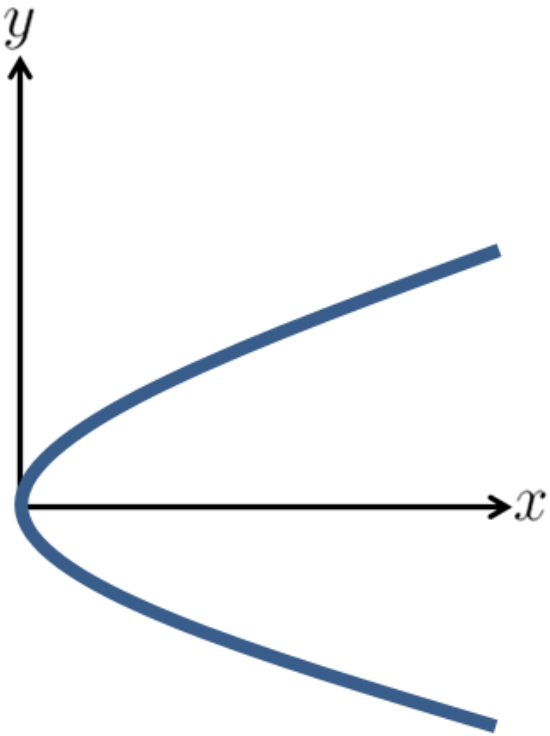
$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

Axial Radii: a, b

Conic Sections (Parabola)

Recall from Algebra or PreCalculus the **canonical form** of a **parabola**:



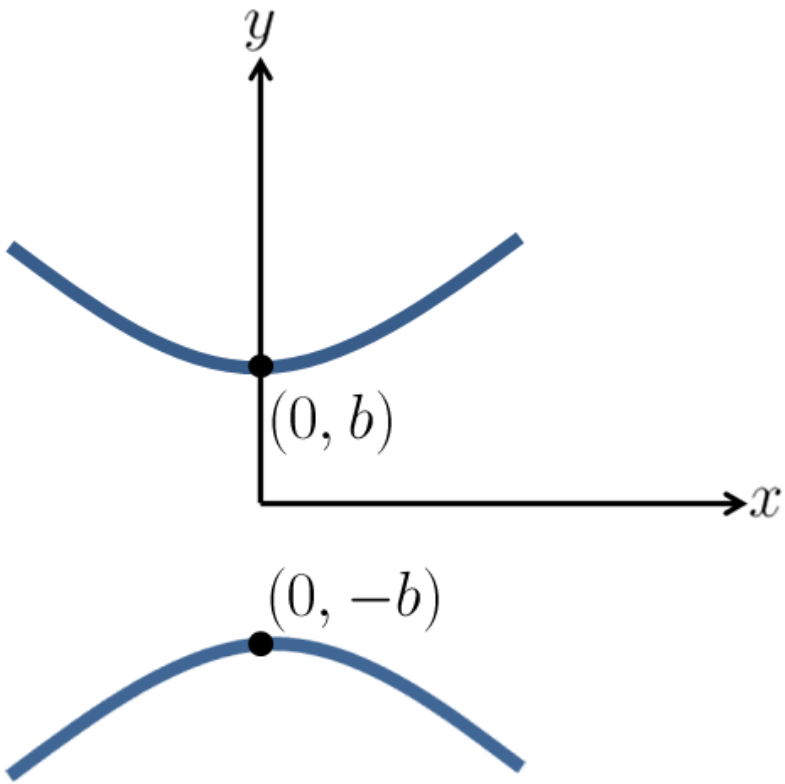


$$y = ax^2 \quad \text{OR} \quad x = by^2$$

Axis of Symmetry: Linear Term

Conic Sections (Hyperbola)

Recall from Algebra or PreCalculus the **canonical form** of a **hyperbola**:

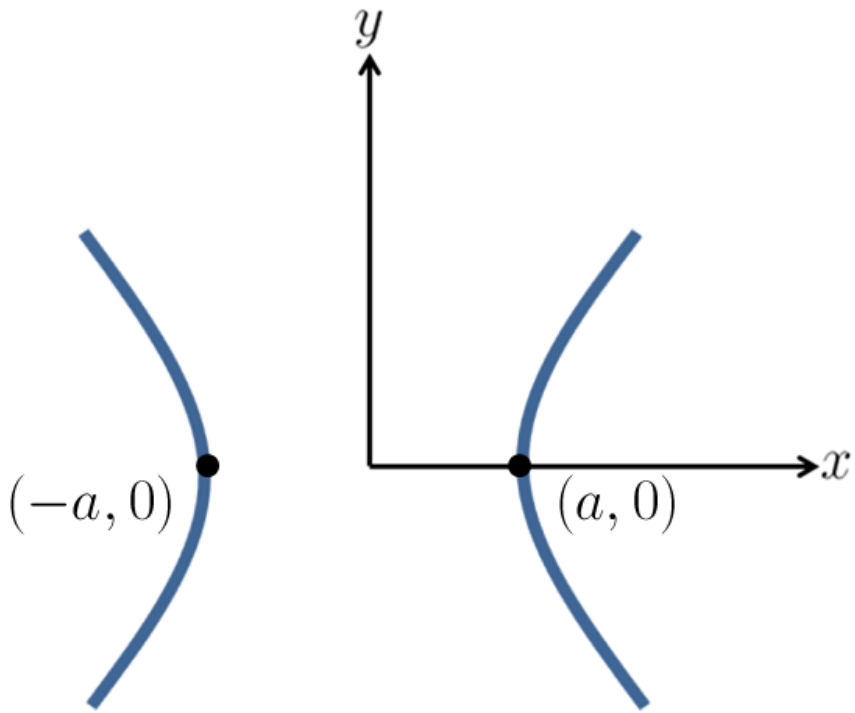


$$\frac{y^2}{b^2} - \frac{x^2}{a^2} = 1$$

Axis of Separation: Negative Square Term

Conic Sections (Hyperbola)

Recall from Algebra or PreCalculus the **canonical form** of a **hyperbola**:



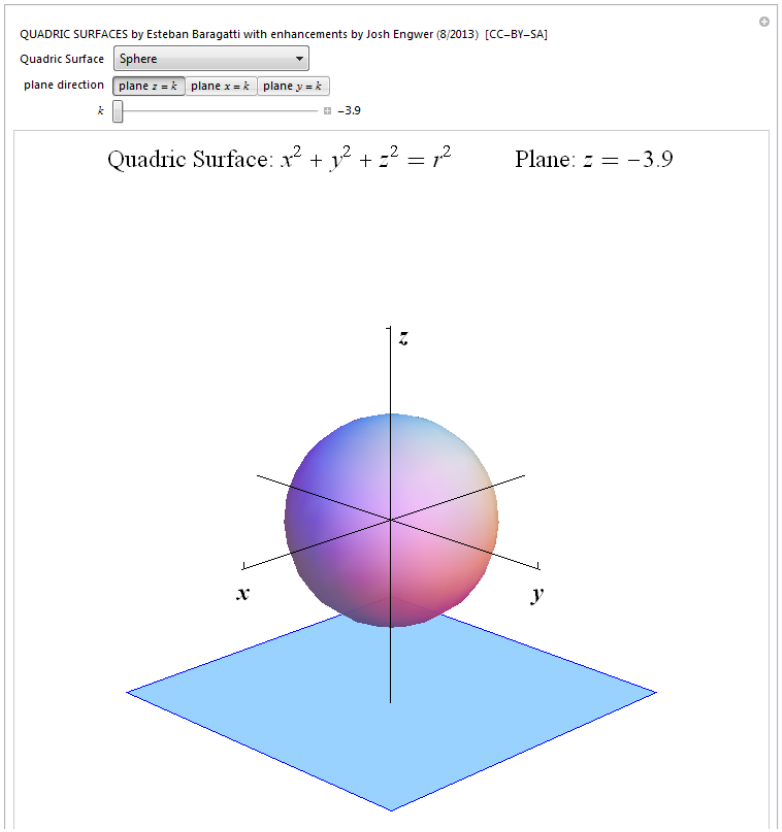
$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

Axis of Separation: Negative Square Term

Quadric Surfaces

- General Form: $Ax^2 + By^2 + Cz^2 + Dxy + Exz + Fyz + Gx + Hy + Iz + J = 0$
- The general form is much too general!
  So, use the canonical forms in the below demo:

(DEMO) QUADRIC SURFACES (click below):



Quadric Surface Identification (Canonical Forms)

Sphere	$x^2 + y^2 + z^2 = r^2$
Ellipsoid	$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$
Parabolic Cylinder	$y = ax^2$ OR $x = by^2$ $z = by^2$ OR $y = cz^2$ $z = ax^2$ OR $x = cz^2$
Elliptic Cylinder	$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ $\frac{x^2}{a^2} + \frac{z^2}{c^2} = 1$ $\frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$
Hyperbolic Cylinder	$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ OR $\frac{y^2}{b^2} - \frac{x^2}{a^2} = 1$ $\frac{x^2}{a^2} - \frac{z^2}{c^2} = 1$ OR $\frac{z^2}{c^2} - \frac{x^2}{a^2} = 1$ $\frac{y^2}{b^2} - \frac{z^2}{c^2} = 1$ OR $\frac{z^2}{c^2} - \frac{y^2}{b^2} = 1$

REMARK: All quadric surfaces are centered at the origin $(0, 0, 0)$.

Quadric Surface Identification (Canonical Forms)

(Elliptic) Paraboloid	$z = \frac{x^2}{a^2} + \frac{y^2}{b^2}$ $y = \frac{x^2}{a^2} + \frac{z^2}{c^2}$ $x = \frac{y^2}{b^2} + \frac{z^2}{c^2}$
(Elliptic) Cone	$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 0$ $\frac{x^2}{a^2} - \frac{y^2}{b^2} + \frac{z^2}{c^2} = 0$ $-\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 0$
Hyperboloid of One Sheet	$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1$ $\frac{x^2}{a^2} - \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$ $-\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$
Hyperboloid of Two Sheets	$-\frac{x^2}{a^2} - \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$ $-\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1$ $\frac{x^2}{a^2} - \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1$

REMARK: All quadric surfaces are centered at the origin $(0, 0, 0)$.

Quadric Surface Identification (Canonical Forms)

Hyperbolic Paraboloid	$z = \frac{y^2}{b^2} - \frac{x^2}{a^2}$	OR	$z = \frac{x^2}{a^2} - \frac{y^2}{b^2}$
	$y = \frac{x^2}{a^2} - \frac{z^2}{c^2}$	OR	$y = \frac{z^2}{c^2} - \frac{x^2}{a^2}$
	$x = \frac{z^2}{c^2} - \frac{y^2}{b^2}$	OR	$x = \frac{y^2}{b^2} - \frac{z^2}{c^2}$

REMARK: All quadric surfaces are centered at the origin $(0, 0, 0)$.

Producing the Canonical Form (Heuristic)

One side of the equation should only have **squared-variable terms**. (x^2, y^2, z^2)
The other side of the equation should only have exactly one of the following:

- Exactly One Linear Variable (x, y, or z)
- The Constant 0
- The Constant 1
- Some other positive constant (i.e. r^2 for a Sphere)

REMARK: The squared-variable terms may have coefficients (e.g. $3x^2$, $\frac{z^2}{7}$)

Quadric Surface Identification (Heuristic)

1^{st}, collect & isolate the squared-variable terms.

2^{nd}, produce the **canonical form**. (see previous slide)

- Constant Term **Zero**? $\xrightarrow{yes}$ (Elliptic) Cone $\rightarrow$ DONE!
- Constant $\neq 0$ & $\neq 1$? $\xrightarrow{yes}$ Sphere $\rightarrow$ DONE!
- 3 Positive Sq-Vars? $\xrightarrow{yes}$ Ellipsoid $\rightarrow$ DONE!
- 2 Negative Sq-Vars? $\xrightarrow{yes}$ Hyperboloid of 2 Sheets $\rightarrow$ DONE!

- Missing Variable? $\xrightarrow{yes}$ Think "Cylinder"
- Linear Variable? $\xrightarrow{yes}$ Think "Parabolic"/"Paraboloid"
- 1 Negative Sq-Var? $\xrightarrow{yes}$ Think "Hyperbolic"/"Hyperboloid of 1 Sheet"
- 2 Positive Sq-Vars? $\xrightarrow{yes}$ Think "Elliptic"

Combine candidate phrases together in a sensible way:

"Cylinder" + "Parabolic"/"Paraboloid" = Parabolic Cylinder

"Elliptic" + "Parabolic"/"Paraboloid" = Elliptic Paraboloid

"Elliptic" + "Hyperbolic"/"Hyperboloid of 1 Sheet" = Hyperboloid of 1 Sheet

Quadric Surface Characterization (Key Properties)

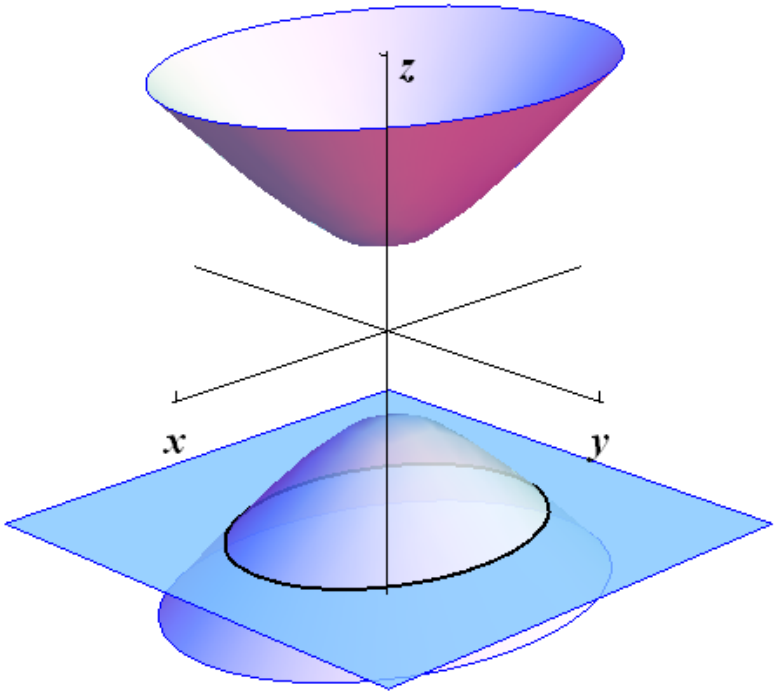
QUADRIC SURFACE	SAMPLE FORM	KEY PROPERTIES
Sphere	$x^2 + y^2 + z^2 = r^2$	Radius: r
Ellipsoid	$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$	Axial Radii: a, b, c
Parabolic Cylinder	$y = ax^2$	Axis of Generation: Omitted variable
Elliptic Cylinder	$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$	Axis of Generation: Omitted variable
Hyperbolic Cylinder	$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$	Axis of Generation: Omitted variable

Quadric Surface Characterization (Key Properties)

QUADRIC SURFACE	SAMPLE FORM	KEY PROPERTIES
(Elliptic) Paraboloid	$z = \frac{x^2}{a^2} + \frac{y^2}{b^2}$	Axis of Revolution: Linear term
(Elliptic) Cone	$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 0$	Axis of Revolution: Negative Square term
Hyperboloid of One Sheet	$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1$	Axis of Revolution: Negative Square term
Hyperboloid of Two Sheets	$-\frac{x^2}{a^2} - \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$	Axis of Revolution: Positive Square term
Hyperbolic Paraboloid	$z = \frac{y^2}{b^2} - \frac{x^2}{a^2}$	(none)

PART II:
QUADRIC SURFACES (CROSS SECTIONS)

Quadric Surfaces (Cross Sections)



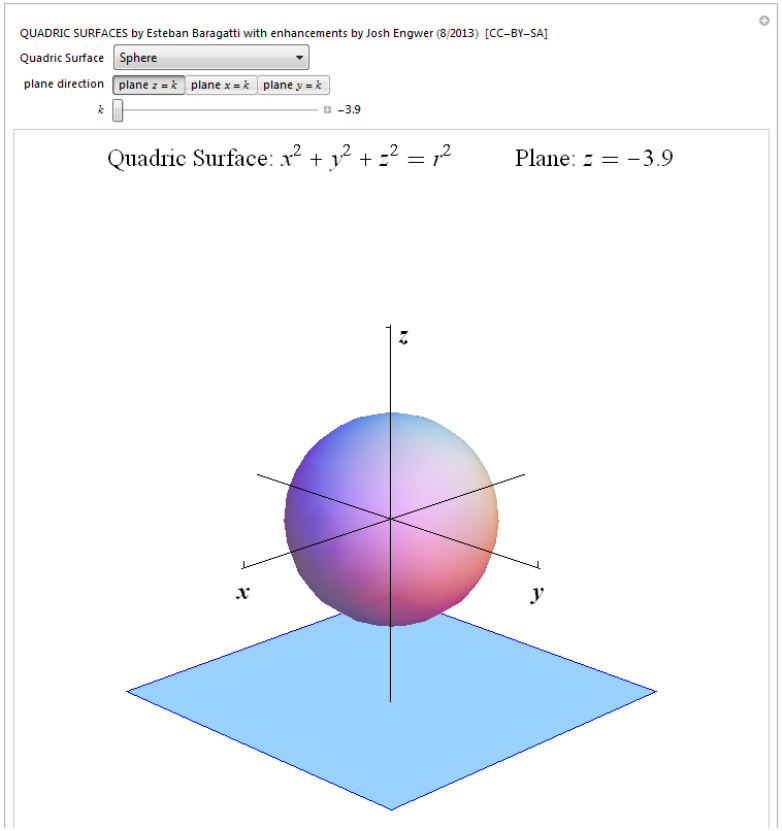
Definition

A **cross section** of a **quadric surface** is the **intersection** of the quadric surface with one of the planes $x = k$, $y = k$, or $z = k$.

REMARK: Often, I'll say "intersection" not "cross section."

Quadric Surfaces (Cross Sections DEMO)

(DEMO) CROSS SECTIONS OF QUADRIC SURFACES (click below):



Fin

Fin.