Multivariable Chain Rules: 2^{nd}-order Derivatives

Calculus III

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TTU

10 December 2014

2^{nd}-order Derivatives using Multivariable Chain Rules (Toolkit)

RESULT	SAMPLE STATEMENT
(D2D) Definition of 2^{nd}-order Derivative	$\frac{d^2w}{dt^2} = \frac{d}{dt}\left[\frac{dw}{dt}\right]$
(D2P) Definition of 2^{nd}-order Partial	$\frac{\partial^2 w}{\partial t^2} = \frac{\partial}{\partial t}\left[\frac{\partial w}{\partial t}\right], \quad \frac{\partial^2 w}{\partial s \partial t} = \frac{\partial}{\partial s}\left[\frac{\partial w}{\partial t}\right]$
(EMP) Equality of Mixed 2^{nd}-order Partials	$\frac{\partial^2 w}{\partial u \partial v} = \frac{\partial^2 w}{\partial v \partial u}$
(PR) Product Rule for Derivatives	$\frac{d}{dt}\left[f(t)g(t)\right] = f'(t)g(t) + f(t)g'(t)$

1-2 Chain Rule

Proposition

(1-2 Chain Rule)

Let $z = f(x) \in C^2$ *where* $x = g(s,t) \in C^{(2,2)}$. *Then:*

$$\frac{\partial z}{\partial s} = \frac{dz}{dx}\frac{\partial x}{\partial s} \qquad \frac{\partial z}{\partial t} = \frac{dz}{dx}\frac{\partial x}{\partial t}$$

"1-2" means 1 **intermediate variable** (x) and 2 **independent var's** (s,t).

How to compute the 2^{nd}**-order partial**: $\dfrac{\partial^2 z}{\partial t^2}$??

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1^{st}, realize that $\frac{\partial z}{\partial t}$, $\frac{dz}{dx}$ **still depend on the intermediate variable** (x).

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$$\frac{\partial^2 z}{\partial t^2} = \frac{\partial}{\partial t}\left[\frac{\partial z}{\partial t}\right]$$

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$$\begin{aligned} \frac{\partial^2 z}{\partial t^2} &= \frac{\partial}{\partial t}\left[\frac{\partial z}{\partial t}\right] \\ &= \frac{\partial}{\partial t}\left[\frac{dz}{dx}\cdot\frac{\partial x}{\partial t}\right] \end{aligned}$$

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&\overset{1-2}{=} \left(\frac{d}{dx}\left[\frac{dz}{dx}\right]\frac{\partial x}{\partial t}\right)\cdot\frac{\partial x}{\partial t} + \frac{dz}{dx}\frac{\partial^2 x}{\partial t^2}
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$$\therefore \quad \boxed{\frac{\partial^2 z}{\partial s \partial t} = \frac{d^2 z}{dx^2}\frac{\partial x}{\partial s}\frac{\partial x}{\partial t} + \frac{dz}{dx}\frac{\partial^2 x}{\partial s \partial t}}$$

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EXERCISE: Starting from scratch, show that

$$\frac{\partial^2 z}{\partial s^2} = \frac{d^2 z}{dx^2}\left(\frac{\partial x}{\partial s}\right)^2 + \frac{dz}{dx}\frac{\partial^2 x}{\partial s^2}$$

2-1 Chain Rule

Proposition

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Let $z = f(x, y) \in C^{(2,2)}$ *where* $x = g(t) \in C^2$ *and* $y = h(t) \in C^2$*. Then:*

$$\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}$$

"2-1" means 2 **intermediate var's** (x, y) and 1 **independent variable** (t).

How to compute the 2^{nd}**-order derivative**: $\dfrac{d^2z}{dt^2}$??

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Let $z = f(x,y) \in C^{(2,2)}$ *where* $x = g(t) \in C^2$ *and* $y = h(t) \in C^2$. *Then:*

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1^{st}, realize that $\frac{dz}{dt}$, $\frac{\partial z}{\partial x}$, $\frac{\partial z}{\partial y}$ **still depend on <u>all</u> intermediate variables** (x,y).

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$$\frac{d^2z}{dt^2} = \frac{d}{dt}\left[\frac{\partial z}{\partial x}\cdot\frac{dx}{dt}\right] + \frac{d}{dt}\left[\frac{\partial z}{\partial y}\cdot\frac{dy}{dt}\right]$$

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&\overset{D2D}{=} \frac{d}{dt}\left[\frac{\partial z}{\partial x}\right] \cdot \frac{dx}{dt} + \frac{\partial z}{\partial x}\frac{d^2x}{dt^2} + \frac{d}{dt}\left[\frac{\partial z}{\partial y}\right] \cdot \frac{dy}{dt} + \frac{\partial z}{\partial y}\frac{d^2y}{dt^2}
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&\overset{2-1}{=} \left(\frac{\partial}{\partial x}\left[\frac{\partial z}{\partial x}\right]\frac{dx}{dt} + \frac{\partial}{\partial y}\left[\frac{\partial z}{\partial x}\right]\frac{dy}{dt}\right)\frac{dx}{dt} + \left(\frac{\partial}{\partial x}\left[\frac{\partial z}{\partial y}\right]\frac{dx}{dt} + \frac{\partial}{\partial y}\left[\frac{\partial z}{\partial y}\right]\frac{dy}{dt}\right)\frac{dy}{dt} + \frac{\partial z}{\partial x}\frac{d^2x}{dt^2} + \frac{\partial z}{\partial y}\frac{d^2y}{dt^2}
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Let $z = f(x,y) \in C^{(2,2)}$ *where* $x = g(t) \in C^2$ *and* $y = h(t) \in C^2$. *Then:*

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&\overset{D2D}{=} \frac{d}{dt}\left[\frac{\partial z}{\partial x}\right]\cdot\frac{dx}{dt} + \frac{\partial z}{\partial x}\frac{d^2x}{dt^2} + \frac{d}{dt}\left[\frac{\partial z}{\partial y}\right]\cdot\frac{dy}{dt} + \frac{\partial z}{\partial y}\frac{d^2y}{dt^2} \\
&\overset{2-1}{=} \left(\frac{\partial}{\partial x}\left[\frac{\partial z}{\partial x}\right]\frac{dx}{dt} + \frac{\partial}{\partial y}\left[\frac{\partial z}{\partial x}\right]\frac{dy}{dt}\right)\frac{dx}{dt} + \left(\frac{\partial}{\partial x}\left[\frac{\partial z}{\partial y}\right]\frac{dx}{dt} + \frac{\partial}{\partial y}\left[\frac{\partial z}{\partial y}\right]\frac{dy}{dt}\right)\frac{dy}{dt} + \frac{\partial z}{\partial x}\frac{d^2x}{dt^2} + \frac{\partial z}{\partial y}\frac{d^2y}{dt^2} \\
&\overset{D2P}{=} \left(\frac{\partial^2 z}{\partial x^2}\frac{dx}{dt} + \frac{\partial^2 z}{\partial y\partial x}\frac{dy}{dt}\right)\frac{dx}{dt} + \left(\frac{\partial^2 z}{\partial x\partial y}\frac{dx}{dt} + \frac{\partial^2 z}{\partial y^2}\frac{dy}{dt}\right)\frac{dy}{dt} + \frac{\partial z}{\partial x}\frac{d^2x}{dt^2} + \frac{\partial z}{\partial y}\frac{d^2y}{dt^2}
\end{aligned}$$

Proposition

(2-1 Chain Rule)

Let $z = f(x,y) \in C^{(2,2)}$ *where* $x = g(t) \in C^2$ *and* $y = h(t) \in C^2$*. Then:*

$$\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}$$

$$\begin{aligned}
\frac{d^2z}{dt^2} &= \frac{d}{dt}\left[\frac{\partial z}{\partial x}\cdot\frac{dx}{dt}\right] + \frac{d}{dt}\left[\frac{\partial z}{\partial y}\cdot\frac{dy}{dt}\right] \\
&\overset{PR}{=} \left(\frac{d}{dt}\left[\frac{\partial z}{\partial x}\right]\cdot\frac{dx}{dt} + \frac{\partial z}{\partial x}\cdot\frac{d}{dt}\left[\frac{dx}{dt}\right]\right) + \left(\frac{d}{dt}\left[\frac{\partial z}{\partial y}\right]\cdot\frac{dy}{dt} + \frac{\partial z}{\partial y}\cdot\frac{d}{dt}\left[\frac{dy}{dt}\right]\right) \\
&\overset{D2D}{=} \frac{d}{dt}\left[\frac{\partial z}{\partial x}\right]\cdot\frac{dx}{dt} + \frac{\partial z}{\partial x}\frac{d^2x}{dt^2} + \frac{d}{dt}\left[\frac{\partial z}{\partial y}\right]\cdot\frac{dy}{dt} + \frac{\partial z}{\partial y}\frac{d^2y}{dt^2} \\
&\overset{2-1}{=} \left(\frac{\partial}{\partial x}\left[\frac{\partial z}{\partial x}\right]\frac{dx}{dt} + \frac{\partial}{\partial y}\left[\frac{\partial z}{\partial x}\right]\frac{dy}{dt}\right)\frac{dx}{dt} + \left(\frac{\partial}{\partial x}\left[\frac{\partial z}{\partial y}\right]\frac{dx}{dt} + \frac{\partial}{\partial y}\left[\frac{\partial z}{\partial y}\right]\frac{dy}{dt}\right)\frac{dy}{dt} + \frac{\partial z}{\partial x}\frac{d^2x}{dt^2} + \frac{\partial z}{\partial y}\frac{d^2y}{dt^2} \\
&\overset{D2P}{=} \left(\frac{\partial^2 z}{\partial x^2}\frac{dx}{dt} + \frac{\partial^2 z}{\partial y\partial x}\frac{dy}{dt}\right)\frac{dx}{dt} + \left(\frac{\partial^2 z}{\partial x\partial y}\frac{dx}{dt} + \frac{\partial^2 z}{\partial y^2}\frac{dy}{dt}\right)\frac{dy}{dt} + \frac{\partial z}{\partial x}\frac{d^2x}{dt^2} + \frac{\partial z}{\partial y}\frac{d^2y}{dt^2} \\
&= \frac{\partial^2 z}{\partial x^2}\left(\frac{dx}{dt}\right)^2 + \frac{\partial^2 z}{\partial y\partial x}\frac{dx}{dt}\frac{dy}{dt} + \frac{\partial^2 z}{\partial x\partial y}\frac{dx}{dt}\frac{dy}{dt} + \frac{\partial^2 z}{\partial y^2}\left(\frac{dy}{dt}\right)^2 + \frac{\partial z}{\partial x}\frac{d^2x}{dt^2} + \frac{\partial z}{\partial y}\frac{d^2y}{dt^2}
\end{aligned}$$

2-1 Chain Rule

Proposition

(2-1 Chain Rule)

Let $z=f(x,y)\in C^{(2,2)}$ where $x=g(t)\in C^2$ and $y=h(t)\in C^2$. Then:

$$\frac{dz}{dt}=\frac{\partial z}{\partial x}\frac{dx}{dt}+\frac{\partial z}{\partial y}\frac{dy}{dt}$$

$$\begin{aligned}
\frac{d^2z}{dt^2} &= \frac{d}{dt}\left[\frac{\partial z}{\partial x}\cdot\frac{dx}{dt}\right]+\frac{d}{dt}\left[\frac{\partial z}{\partial y}\cdot\frac{dy}{dt}\right]\\
&\overset{PR}{=} \left(\frac{d}{dt}\left[\frac{\partial z}{\partial x}\right]\cdot\frac{dx}{dt}+\frac{\partial z}{\partial x}\cdot\frac{d}{dt}\left[\frac{dx}{dt}\right]\right)+\left(\frac{d}{dt}\left[\frac{\partial z}{\partial y}\right]\cdot\frac{dy}{dt}+\frac{\partial z}{\partial y}\cdot\frac{d}{dt}\left[\frac{dy}{dt}\right]\right)\\
&\overset{D2D}{=} \frac{d}{dt}\left[\frac{\partial z}{\partial x}\right]\cdot\frac{dx}{dt}+\frac{\partial z}{\partial x}\frac{d^2x}{dt^2}+\frac{d}{dt}\left[\frac{\partial z}{\partial y}\right]\cdot\frac{dy}{dt}+\frac{\partial z}{\partial y}\frac{d^2y}{dt^2}\\
&\overset{2-1}{=} \left(\frac{\partial}{\partial x}\left[\frac{\partial z}{\partial x}\right]\frac{dx}{dt}+\frac{\partial}{\partial y}\left[\frac{\partial z}{\partial x}\right]\frac{dy}{dt}\right)\frac{dx}{dt}+\left(\frac{\partial}{\partial x}\left[\frac{\partial z}{\partial y}\right]\frac{dx}{dt}+\frac{\partial}{\partial y}\left[\frac{\partial z}{\partial y}\right]\frac{dy}{dt}\right)\frac{dy}{dt}+\frac{\partial z}{\partial x}\frac{d^2x}{dt^2}+\frac{\partial z}{\partial y}\frac{d^2y}{dt^2}\\
&\overset{D2P}{=} \left(\frac{\partial^2 z}{\partial x^2}\frac{dx}{dt}+\frac{\partial^2 z}{\partial y\partial x}\frac{dy}{dt}\right)\frac{dx}{dt}+\left(\frac{\partial^2 z}{\partial x\partial y}\frac{dx}{dt}+\frac{\partial^2 z}{\partial y^2}\frac{dy}{dt}\right)\frac{dy}{dt}+\frac{\partial z}{\partial x}\frac{d^2x}{dt^2}+\frac{\partial z}{\partial y}\frac{d^2y}{dt^2}\\
&= \frac{\partial^2 z}{\partial x^2}\left(\frac{dx}{dt}\right)^2+\frac{\partial^2 z}{\partial y\partial x}\frac{dx}{dt}\frac{dy}{dt}+\frac{\partial^2 z}{\partial x\partial y}\frac{dx}{dt}\frac{dy}{dt}+\frac{\partial^2 z}{\partial y^2}\left(\frac{dy}{dt}\right)^2+\frac{\partial z}{\partial x}\frac{d^2x}{dt^2}+\frac{\partial z}{\partial y}\frac{d^2y}{dt^2}\\
&\overset{EMP}{=} \frac{\partial^2 z}{\partial x^2}\left(\frac{dx}{dt}\right)^2+2\frac{\partial^2 z}{\partial x\partial y}\frac{dx}{dt}\frac{dy}{dt}+\frac{\partial^2 z}{\partial y^2}\left(\frac{dy}{dt}\right)^2+\frac{\partial z}{\partial x}\frac{d^2x}{dt^2}+\frac{\partial z}{\partial y}\frac{d^2y}{dt^2}
\end{aligned}$$

2-1 Chain Rule

Proposition

(2-1 Chain Rule)

Let $z = f(x,y) \in C^{(2,2)}$ *where* $x = g(t) \in C^2$ *and* $y = h(t) \in C^2$. *Then:*

$$\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}$$

$$\therefore \quad \boxed{\frac{d^2z}{dt^2} = \frac{\partial^2 z}{\partial x^2}\left(\frac{dx}{dt}\right)^2 + 2\frac{\partial^2 z}{\partial x \partial y}\frac{dx}{dt}\frac{dy}{dt} + \frac{\partial^2 z}{\partial y^2}\left(\frac{dy}{dt}\right)^2 + \frac{\partial z}{\partial x}\frac{d^2x}{dt^2} + \frac{\partial z}{\partial y}\frac{d^2y}{dt^2}}$$

2-2 Chain Rule

Proposition

(2-2 Chain Rule)

Let $z = f(x,y) \in C^{(2,2)}$ *where* $x = g(s,t) \in C^{(2,2)}$ *and* $y = h(s,t) \in C^{(2,2)}$*. Then:*

$$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial s} \qquad \frac{\partial z}{\partial t} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial t}$$

"2-2" means 2 **intermediate var's** (x,y) and 2 **independent var's** (s,t).

How to compute the 2^{nd}**-order partial**: $\dfrac{\partial^2 z}{\partial t^2}$??

Proposition

(2-2 Chain Rule)

Let $z = f(x,y) \in C^{(2,2)}$ *where* $x = g(s,t) \in C^{(2,2)}$ *and* $y = h(s,t) \in C^{(2,2)}$*. Then:*

$$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial s} \qquad \frac{\partial z}{\partial t} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial t}$$

1^{st}, realize that $\frac{\partial z}{\partial t}, \frac{\partial z}{\partial x}, \frac{\partial z}{\partial y}$ **still depend on all intermediate variables** (x,y).

Proposition

(2-2 Chain Rule)

Let $z = f(x,y) \in C^{(2,2)}$ *where* $x = g(s,t) \in C^{(2,2)}$ *and* $y = h(s,t) \in C^{(2,2)}$. *Then:*

$$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial s} \qquad \frac{\partial z}{\partial t} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial t}$$

1^{st}, realize that $\frac{\partial z}{\partial t}, \frac{\partial z}{\partial x}, \frac{\partial z}{\partial y}$ **still depend on <u>all</u> intermediate variables** (x,y).

$$\frac{\partial^2 z}{\partial t^2} = \frac{\partial}{\partial t}\left[\frac{\partial z}{\partial x} \cdot \frac{\partial x}{\partial t}\right] + \frac{\partial}{\partial t}\left[\frac{\partial z}{\partial y} \cdot \frac{\partial y}{\partial t}\right]$$

2-2 Chain Rule

Proposition

(2-2 Chain Rule)

Let $z = f(x,y) \in C^{(2,2)}$ where $x = g(s,t) \in C^{(2,2)}$ and $y = h(s,t) \in C^{(2,2)}$. Then:

$$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial s} \qquad \frac{\partial z}{\partial t} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial t}$$

1^{st}, realize that $\frac{\partial z}{\partial t}, \frac{\partial z}{\partial x}, \frac{\partial z}{\partial y}$ **still depend on <u>all</u> intermediate variables** (x,y).

$$\begin{aligned} \frac{\partial^2 z}{\partial t^2} &= \frac{\partial}{\partial t}\left[\frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial t}\right] + \frac{\partial}{\partial t}\left[\frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial t}\right] \\ &\overset{PR}{=} \left(\frac{\partial}{\partial t}\left[\frac{\partial z}{\partial x}\right]\cdot\frac{\partial x}{\partial t} + \frac{\partial z}{\partial x}\cdot\frac{\partial}{\partial t}\left[\frac{\partial x}{\partial t}\right]\right) + \left(\frac{\partial}{\partial t}\left[\frac{\partial z}{\partial y}\right]\cdot\frac{\partial y}{\partial t} + \frac{\partial z}{\partial y}\cdot\frac{\partial}{\partial t}\left[\frac{\partial y}{\partial t}\right]\right) \end{aligned}$$

2-2 Chain Rule

Proposition

(2-2 Chain Rule)

Let $z = f(x,y) \in C^{(2,2)}$ *where* $x = g(s,t) \in C^{(2,2)}$ *and* $y = h(s,t) \in C^{(2,2)}$*. Then:*

$$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial s} \qquad \frac{\partial z}{\partial t} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial t}$$

1^{st}, realize that $\frac{\partial z}{\partial t}, \frac{\partial z}{\partial x}, \frac{\partial z}{\partial y}$ **still depend on <u>all</u> intermediate variables** (x,y).

$$\begin{aligned}
\frac{\partial^2 z}{\partial t^2} &= \frac{\partial}{\partial t}\left[\frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial t}\right] + \frac{\partial}{\partial t}\left[\frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial t}\right] \\
&\overset{PR}{=} \left(\frac{\partial}{\partial t}\left[\frac{\partial z}{\partial x}\right]\cdot\frac{\partial x}{\partial t} + \frac{\partial z}{\partial x}\cdot\frac{\partial}{\partial t}\left[\frac{\partial x}{\partial t}\right]\right) + \left(\frac{\partial}{\partial t}\left[\frac{\partial z}{\partial y}\right]\cdot\frac{\partial y}{\partial t} + \frac{\partial z}{\partial y}\cdot\frac{\partial}{\partial t}\left[\frac{\partial y}{\partial t}\right]\right) \\
&\overset{D2P}{=} \frac{\partial}{\partial t}\left[\frac{\partial z}{\partial x}\right]\cdot\frac{\partial x}{\partial t} + \frac{\partial z}{\partial x}\frac{\partial^2 x}{\partial t^2} + \frac{\partial}{\partial t}\left[\frac{\partial z}{\partial y}\right]\cdot\frac{\partial y}{\partial t} + \frac{\partial z}{\partial y}\frac{\partial^2 y}{\partial t^2}
\end{aligned}$$

Proposition

(2-2 Chain Rule)

Let $z = f(x,y) \in C^{(2,2)}$ *where* $x = g(s,t) \in C^{(2,2)}$ *and* $y = h(s,t) \in C^{(2,2)}$. *Then:*

$$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial s} \qquad \frac{\partial z}{\partial t} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial t}$$

1^{st}, realize that $\frac{\partial z}{\partial t}, \frac{\partial z}{\partial x}, \frac{\partial z}{\partial y}$ **still depend on <u>all</u> intermediate variables** (x,y).

$$\begin{aligned}
\frac{\partial^2 z}{\partial t^2} &= \frac{\partial}{\partial t}\left[\frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial t}\right] + \frac{\partial}{\partial t}\left[\frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial t}\right] \\
&\overset{PR}{=} \left(\frac{\partial}{\partial t}\left[\frac{\partial z}{\partial x}\right]\cdot\frac{\partial x}{\partial t} + \frac{\partial z}{\partial x}\cdot\frac{\partial}{\partial t}\left[\frac{\partial x}{\partial t}\right]\right) + \left(\frac{\partial}{\partial t}\left[\frac{\partial z}{\partial y}\right]\cdot\frac{\partial y}{\partial t} + \frac{\partial z}{\partial y}\cdot\frac{\partial}{\partial t}\left[\frac{\partial y}{\partial t}\right]\right) \\
&\overset{D2P}{=} \frac{\partial}{\partial t}\left[\frac{\partial z}{\partial x}\right]\cdot\frac{\partial x}{\partial t} + \frac{\partial z}{\partial x}\frac{\partial^2 x}{\partial t^2} + \frac{\partial}{\partial t}\left[\frac{\partial z}{\partial y}\right]\cdot\frac{\partial y}{\partial t} + \frac{\partial z}{\partial y}\frac{\partial^2 y}{\partial t^2} \\
&\overset{2-2}{=} \left(\frac{\partial}{\partial x}\left[\frac{\partial z}{\partial x}\right]\frac{\partial x}{\partial t} + \frac{\partial}{\partial y}\left[\frac{\partial z}{\partial x}\right]\frac{\partial y}{\partial t}\right)\frac{\partial x}{\partial t} + \left(\frac{\partial}{\partial x}\left[\frac{\partial z}{\partial y}\right]\frac{\partial x}{\partial t} + \frac{\partial}{\partial y}\left[\frac{\partial z}{\partial y}\right]\frac{\partial y}{\partial t}\right)\frac{\partial y}{\partial t} + \frac{\partial z}{\partial x}\frac{\partial^2 x}{\partial t^2} + \frac{\partial z}{\partial y}\frac{\partial^2 y}{\partial t^2}
\end{aligned}$$

Proposition

(2-2 Chain Rule)

Let $z = f(x,y) \in C^{(2,2)}$ *where* $x = g(s,t) \in C^{(2,2)}$ *and* $y = h(s,t) \in C^{(2,2)}$. *Then:*

$$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial s} \qquad \frac{\partial z}{\partial t} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial t}$$

$$\begin{aligned}
\frac{\partial^2 z}{\partial t^2} &= \frac{\partial}{\partial t}\left[\frac{\partial z}{\partial x} \cdot \frac{\partial x}{\partial t}\right] + \frac{\partial}{\partial t}\left[\frac{\partial z}{\partial y} \cdot \frac{\partial y}{\partial t}\right] \\
&\overset{PR}{=} \left(\frac{\partial}{\partial t}\left[\frac{\partial z}{\partial x}\right] \cdot \frac{\partial x}{\partial t} + \frac{\partial z}{\partial x} \cdot \frac{\partial}{\partial t}\left[\frac{\partial x}{\partial t}\right]\right) + \left(\frac{\partial}{\partial t}\left[\frac{\partial z}{\partial y}\right] \cdot \frac{\partial y}{\partial t} + \frac{\partial z}{\partial y} \cdot \frac{\partial}{\partial t}\left[\frac{\partial y}{\partial t}\right]\right) \\
&\overset{D2P}{=} \frac{\partial}{\partial t}\left[\frac{\partial z}{\partial x}\right] \cdot \frac{\partial x}{\partial t} + \frac{\partial z}{\partial x}\frac{\partial^2 x}{\partial t^2} + \frac{\partial}{\partial t}\left[\frac{\partial z}{\partial y}\right] \cdot \frac{\partial y}{\partial t} + \frac{\partial z}{\partial y}\frac{\partial^2 y}{\partial t^2} \\
&\overset{2-2}{=} \left(\frac{\partial}{\partial x}\left[\frac{\partial z}{\partial x}\right]\frac{\partial x}{\partial t} + \frac{\partial}{\partial y}\left[\frac{\partial z}{\partial x}\right]\frac{\partial y}{\partial t}\right)\frac{\partial x}{\partial t} + \left(\frac{\partial}{\partial x}\left[\frac{\partial z}{\partial y}\right]\frac{\partial x}{\partial t} + \frac{\partial}{\partial y}\left[\frac{\partial z}{\partial y}\right]\frac{\partial y}{\partial t}\right)\frac{\partial y}{\partial t} + \frac{\partial z}{\partial x}\frac{\partial^2 x}{\partial t^2} + \frac{\partial z}{\partial y} \cdots \\
&\overset{D2P}{=} \left(\frac{\partial^2 z}{\partial x^2}\frac{\partial x}{\partial t} + \frac{\partial^2 z}{\partial y \partial x}\frac{\partial y}{\partial t}\right)\frac{\partial x}{\partial t} + \left(\frac{\partial^2 z}{\partial x \partial y}\frac{\partial x}{\partial t} + \frac{\partial^2 z}{\partial y^2}\frac{\partial y}{\partial t}\right)\frac{\partial y}{\partial t} + \frac{\partial z}{\partial x}\frac{\partial^2 x}{\partial t^2} + \frac{\partial z}{\partial y}\frac{\partial^2 y}{\partial t^2}
\end{aligned}$$

2-2 Chain Rule

Proposition

(2-2 Chain Rule)

Let $z = f(x,y) \in C^{(2,2)}$ *where* $x = g(s,t) \in C^{(2,2)}$ *and* $y = h(s,t) \in C^{(2,2)}$*. Then:*

$$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial s} \qquad \frac{\partial z}{\partial t} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial t}$$

$$\begin{aligned}
\frac{\partial^2 z}{\partial t^2} &= \frac{\partial}{\partial t}\left[\frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial t}\right] + \frac{\partial}{\partial t}\left[\frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial t}\right] \\
&\overset{PR}{=} \left(\frac{\partial}{\partial t}\left[\frac{\partial z}{\partial x}\right]\cdot\frac{\partial x}{\partial t} + \frac{\partial z}{\partial x}\cdot\frac{\partial}{\partial t}\left[\frac{\partial x}{\partial t}\right]\right) + \left(\frac{\partial}{\partial t}\left[\frac{\partial z}{\partial y}\right]\cdot\frac{\partial y}{\partial t} + \frac{\partial z}{\partial y}\cdot\frac{\partial}{\partial t}\left[\frac{\partial y}{\partial t}\right]\right) \\
&\overset{D2P}{=} \frac{\partial}{\partial t}\left[\frac{\partial z}{\partial x}\right]\cdot\frac{\partial x}{\partial t} + \frac{\partial z}{\partial x}\frac{\partial^2 x}{\partial t^2} + \frac{\partial}{\partial t}\left[\frac{\partial z}{\partial y}\right]\cdot\frac{\partial y}{\partial t} + \frac{\partial z}{\partial y}\frac{\partial^2 y}{\partial t^2} \\
&\overset{2-2}{=} \left(\frac{\partial}{\partial x}\left[\frac{\partial z}{\partial x}\right]\frac{\partial x}{\partial t} + \frac{\partial}{\partial y}\left[\frac{\partial z}{\partial x}\right]\frac{\partial y}{\partial t}\right)\frac{\partial x}{\partial t} + \left(\frac{\partial}{\partial x}\left[\frac{\partial z}{\partial y}\right]\frac{\partial x}{\partial t} + \frac{\partial}{\partial y}\left[\frac{\partial z}{\partial y}\right]\frac{\partial y}{\partial t}\right)\frac{\partial y}{\partial t} + \frac{\partial z}{\partial x}\frac{\partial^2 x}{\partial t^2} + \frac{\partial z}{\partial y}\,\partial \\
&\overset{D2P}{=} \left(\frac{\partial^2 z}{\partial x^2}\frac{\partial x}{\partial t} + \frac{\partial^2 z}{\partial y \partial x}\frac{\partial y}{\partial t}\right)\frac{\partial x}{\partial t} + \left(\frac{\partial^2 z}{\partial x \partial y}\frac{\partial x}{\partial t} + \frac{\partial^2 z}{\partial y^2}\frac{\partial y}{\partial t}\right)\frac{\partial y}{\partial t} + \frac{\partial z}{\partial x}\frac{\partial^2 x}{\partial t^2} + \frac{\partial z}{\partial y}\frac{\partial^2 y}{\partial t^2} \\
&= \frac{\partial^2 z}{\partial x^2}\left(\frac{\partial x}{\partial t}\right)^2 + \frac{\partial^2 z}{\partial y \partial x}\frac{\partial x}{\partial t}\frac{\partial y}{\partial t} + \frac{\partial^2 z}{\partial x \partial y}\frac{\partial x}{\partial t}\frac{\partial y}{\partial t} + \frac{\partial^2 z}{\partial y^2}\left(\frac{\partial y}{\partial t}\right)^2 + \frac{\partial z}{\partial x}\frac{\partial^2 x}{\partial t^2} + \frac{\partial z}{\partial y}\frac{\partial^2 y}{\partial t^2}
\end{aligned}$$

Proposition

(2-2 Chain Rule)

Let $z = f(x, y) \in C^{(2,2)}$ *where* $x = g(s, t) \in C^{(2,2)}$ *and* $y = h(s, t) \in C^{(2,2)}$. *Then:*

$$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial s} \qquad \frac{\partial z}{\partial t} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial t}$$

$$\begin{aligned}
\frac{\partial^2 z}{\partial t^2} &= \frac{\partial}{\partial t}\left[\frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial t}\right] + \frac{\partial}{\partial t}\left[\frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial t}\right] \\
&\overset{PR}{=} \left(\frac{\partial}{\partial t}\left[\frac{\partial z}{\partial x}\right]\cdot\frac{\partial x}{\partial t} + \frac{\partial z}{\partial x}\cdot\frac{\partial}{\partial t}\left[\frac{\partial x}{\partial t}\right]\right) + \left(\frac{\partial}{\partial t}\left[\frac{\partial z}{\partial y}\right]\cdot\frac{\partial y}{\partial t} + \frac{\partial z}{\partial y}\cdot\frac{\partial}{\partial t}\left[\frac{\partial y}{\partial t}\right]\right) \\
&\overset{D2P}{=} \frac{\partial}{\partial t}\left[\frac{\partial z}{\partial x}\right]\cdot\frac{\partial x}{\partial t} + \frac{\partial z}{\partial x}\frac{\partial^2 x}{\partial t^2} + \frac{\partial}{\partial t}\left[\frac{\partial z}{\partial y}\right]\cdot\frac{\partial y}{\partial t} + \frac{\partial z}{\partial y}\frac{\partial^2 y}{\partial t^2} \\
&\overset{2-2}{=} \left(\frac{\partial}{\partial x}\left[\frac{\partial z}{\partial x}\right]\frac{\partial x}{\partial t} + \frac{\partial}{\partial y}\left[\frac{\partial z}{\partial x}\right]\frac{\partial y}{\partial t}\right)\frac{\partial x}{\partial t} + \left(\frac{\partial}{\partial x}\left[\frac{\partial z}{\partial y}\right]\frac{\partial x}{\partial t} + \frac{\partial}{\partial y}\left[\frac{\partial z}{\partial y}\right]\frac{\partial y}{\partial t}\right)\frac{\partial y}{\partial t} + \frac{\partial z}{\partial x}\frac{\partial^2 x}{\partial t^2} + \frac{\partial z}{\partial y} \\
&\overset{D2P}{=} \left(\frac{\partial^2 z}{\partial x^2}\frac{\partial x}{\partial t} + \frac{\partial^2 z}{\partial y \partial x}\frac{\partial y}{\partial t}\right)\frac{\partial x}{\partial t} + \left(\frac{\partial^2 z}{\partial x \partial y}\frac{\partial x}{\partial t} + \frac{\partial^2 z}{\partial y^2}\frac{\partial y}{\partial t}\right)\frac{\partial y}{\partial t} + \frac{\partial z}{\partial x}\frac{\partial^2 x}{\partial t^2} + \frac{\partial z}{\partial y}\frac{\partial^2 y}{\partial t^2} \\
&= \frac{\partial^2 z}{\partial x^2}\left(\frac{\partial x}{\partial t}\right)^2 + \frac{\partial^2 z}{\partial y \partial x}\frac{\partial x}{\partial t}\frac{\partial y}{\partial t} + \frac{\partial^2 z}{\partial x \partial y}\frac{\partial x}{\partial t}\frac{\partial y}{\partial t} + \frac{\partial^2 z}{\partial y^2}\left(\frac{\partial y}{\partial t}\right)^2 + \frac{\partial z}{\partial x}\frac{\partial^2 x}{\partial t^2} + \frac{\partial z}{\partial y}\frac{\partial^2 y}{\partial t^2} \\
&\overset{EMP}{=} \frac{\partial^2 z}{\partial x^2}\left(\frac{\partial x}{\partial t}\right)^2 + 2\frac{\partial^2 z}{\partial x \partial y}\frac{\partial x}{\partial t}\frac{\partial y}{\partial t} + \frac{\partial^2 z}{\partial y^2}\left(\frac{\partial y}{\partial t}\right)^2 + \frac{\partial z}{\partial x}\frac{\partial^2 x}{\partial t^2} + \frac{\partial z}{\partial y}\frac{\partial^2 y}{\partial t^2}
\end{aligned}$$

2-2 Chain Rule

Proposition

(2-2 Chain Rule)

Let $z = f(x, y) \in C^{(2,2)}$ *where* $x = g(s, t) \in C^{(2,2)}$ *and* $y = h(s, t) \in C^{(2,2)}$*. Then:*

$$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial s} \qquad \frac{\partial z}{\partial t} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial t}$$

$$\therefore \quad \boxed{\frac{\partial^2 z}{\partial t^2} = \frac{\partial^2 z}{\partial x^2}\left(\frac{\partial x}{\partial t}\right)^2 + 2\frac{\partial^2 z}{\partial x \partial y}\frac{\partial x}{\partial t}\frac{\partial y}{\partial t} + \frac{\partial^2 z}{\partial y^2}\left(\frac{\partial y}{\partial t}\right)^2 + \frac{\partial z}{\partial x}\frac{\partial^2 x}{\partial t^2} + \frac{\partial z}{\partial y}\frac{\partial^2 y}{\partial t^2}}$$

2-2 Chain Rule

Proposition

(2-2 Chain Rule)

Let $z = f(x,y) \in C^{(2,2)}$ *where* $x = g(s,t) \in C^{(2,2)}$ *and* $y = h(s,t) \in C^{(2,2)}$. *Then:*

$$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial s} \qquad\qquad \frac{\partial z}{\partial t} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial t}$$

EXERCISE: Starting from scratch, show that

$$\frac{\partial^2 z}{\partial s^2} = \frac{\partial^2 z}{\partial x^2}\left(\frac{\partial x}{\partial s}\right)^2 + 2\frac{\partial^2 z}{\partial x \partial y}\frac{\partial x}{\partial s}\frac{\partial y}{\partial s} + \frac{\partial^2 z}{\partial y^2}\left(\frac{\partial y}{\partial s}\right)^2 + \frac{\partial z}{\partial x}\frac{\partial^2 x}{\partial s^2} + \frac{\partial z}{\partial y}\frac{\partial^2 y}{\partial s^2}$$

2-2 Chain Rule

Proposition

(2-2 Chain Rule)

Let $z = f(x,y) \in C^{(2,2)}$ *where* $x = g(s,t) \in C^{(2,2)}$ *and* $y = h(s,t) \in C^{(2,2)}$*. Then:*

$$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial s} \qquad \frac{\partial z}{\partial t} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial t}$$

EXERCISE: Starting from scratch, show that

$$\frac{\partial^2 z}{\partial s \partial t} = \frac{\partial^2 z}{\partial x^2}\frac{\partial x}{\partial s}\frac{\partial x}{\partial t} + \frac{\partial^2 z}{\partial x \partial y}\left(\frac{\partial y}{\partial s}\frac{\partial x}{\partial t} + \frac{\partial x}{\partial s}\frac{\partial y}{\partial t}\right) + \frac{\partial^2 z}{\partial y^2}\frac{\partial y}{\partial s}\frac{\partial y}{\partial t} + \frac{\partial z}{\partial x}\frac{\partial^2 x}{\partial s \partial t} + \frac{\partial z}{\partial y}\frac{\partial^2 y}{\partial s \partial t}$$

Fin

Fin.