The Laplacian in Polar Coordinates

Calculus III

Josh Engwer

TTU

10 December 2014

Converting from Rectangular $\to$ Polar Coordinates

GOAL: Express the **Laplacian** in **polar coordinates**.

PROCEDURE: $\left(u : \mathbb{R}^2 \to \mathbb{R}\right.$ is assumed to be $C^{(2,2)}(D)$, where $\left. D \subseteq \mathbb{R}^2\right)$

1. Recall Laplacian in rectangular coordinates: $\nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}$
2. Recall conversion of rect. coords $\to$ polar coords: $\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases}$
3. Use 2-2 Chain Rule: $\frac{\partial u}{\partial r} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial r}$ $\qquad \frac{\partial u}{\partial \theta} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial \theta} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial \theta}$
4. Express 1^{st}-order partials $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}$ in polar coordinates
5. Express 2^{nd}-order partials $\frac{\partial u^2}{\partial x^2}, \frac{\partial^2 u}{\partial y^2}$ in polar coordinates
6. Express the Laplacian in polar coordinates

WHY DO THIS???

- Scalar field u may be expressed in polar coordinates a priori.
- Region of interest D may be a polar region. (e.g. disk, annulus, ...)

Use the 2-2 Multivariable Chain Rule

$$\nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \qquad \begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases}$$

$$\begin{cases} \dfrac{\partial u}{\partial r} = \dfrac{\partial u}{\partial x}\dfrac{\partial x}{\partial r} + \dfrac{\partial u}{\partial y}\dfrac{\partial y}{\partial r} & = \quad \cos\theta\dfrac{\partial u}{\partial x} + \sin\theta\dfrac{\partial u}{\partial y} \\[2ex] \dfrac{\partial u}{\partial \theta} = \dfrac{\partial u}{\partial x}\dfrac{\partial x}{\partial \theta} + \dfrac{\partial u}{\partial y}\dfrac{\partial y}{\partial \theta} & = \quad -r\sin\theta\dfrac{\partial u}{\partial x} + r\cos\theta\dfrac{\partial u}{\partial y} \end{cases}$$

Solve resulting Linear System for $\dfrac{\partial u}{\partial x}$ & $\dfrac{\partial u}{\partial y}$

$$\nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \qquad \begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases}$$

$$\begin{cases} \dfrac{\partial u}{\partial r} &= & \cos\theta \dfrac{\partial u}{\partial x} &+& \sin\theta \dfrac{\partial u}{\partial y} \\[2ex] \dfrac{\partial u}{\partial \theta} &= & -r\sin\theta \dfrac{\partial u}{\partial x} &+& r\cos\theta \dfrac{\partial u}{\partial y} \end{cases}$$

Solve resulting Linear System for $\dfrac{\partial u}{\partial x}$ & $\dfrac{\partial u}{\partial y}$

$$\nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \qquad \begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases}$$

$$\begin{cases} \dfrac{\partial u}{\partial r} = \cos\theta \dfrac{\partial u}{\partial x} + \sin\theta \dfrac{\partial u}{\partial y} \\[2ex] \dfrac{\partial u}{\partial \theta} = -r\sin\theta \dfrac{\partial u}{\partial x} + r\cos\theta \dfrac{\partial u}{\partial y} \end{cases}$$

Solve resulting Linear System for $\frac{\partial u}{\partial x}$ & $\frac{\partial u}{\partial y}$

$$\nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \qquad \begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases}$$

$$\begin{cases} \dfrac{\partial u}{\partial r} = \cos\theta \dfrac{\partial u}{\partial x} + \sin\theta \dfrac{\partial u}{\partial y} \\[2ex] \dfrac{1}{r}\dfrac{\partial u}{\partial \theta} = -\sin\theta \dfrac{\partial u}{\partial x} + \cos\theta \dfrac{\partial u}{\partial y} \end{cases}$$

Solve resulting Linear System for $\dfrac{\partial u}{\partial x}$ & $\dfrac{\partial u}{\partial y}$

$$\nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \qquad \begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases}$$

$$\begin{cases} \sin\theta \dfrac{\partial u}{\partial r} = \sin\theta\cos\theta \dfrac{\partial u}{\partial x} + \sin^2\theta \dfrac{\partial u}{\partial y} \\ \dfrac{\cos\theta}{r}\dfrac{\partial u}{\partial \theta} = -\cos\theta\sin\theta \dfrac{\partial u}{\partial x} + \cos^2\theta \dfrac{\partial u}{\partial y} \end{cases}$$

Solve resulting Linear System for $\frac{\partial u}{\partial x}$ & $\frac{\partial u}{\partial y}$

$$\nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \qquad \begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases}$$

$$\begin{cases} \sin\theta \frac{\partial u}{\partial r} = \sin\theta\cos\theta \frac{\partial u}{\partial x} + \sin^2\theta \frac{\partial u}{\partial y} \\ \frac{\cos\theta}{r}\frac{\partial u}{\partial \theta} = -\cos\theta\sin\theta \frac{\partial u}{\partial x} + \cos^2\theta \frac{\partial u}{\partial y} \end{cases}$$

Add the two equations:

$$\implies \sin\theta \frac{\partial u}{\partial r} + \frac{\cos\theta}{r}\frac{\partial u}{\partial \theta} = \left(\sin^2\theta + \cos^2\theta\right)\frac{\partial u}{\partial y} = \frac{\partial u}{\partial y}$$

$$\therefore \quad \frac{\partial u}{\partial y} = \sin\theta \frac{\partial u}{\partial r} + \frac{\cos\theta}{r}\frac{\partial u}{\partial \theta}$$

Solve resulting Linear System for $\frac{\partial u}{\partial x}$ & $\frac{\partial u}{\partial y}$

$$\frac{\partial u}{\partial y} = \sin\theta \frac{\partial u}{\partial r} + \frac{\cos\theta}{r} \frac{\partial u}{\partial \theta}$$

$$\begin{cases} \dfrac{\partial u}{\partial r} & = & \cos\theta \dfrac{\partial u}{\partial x} & + & \sin\theta \dfrac{\partial u}{\partial y} \\ \dfrac{1}{r}\dfrac{\partial u}{\partial \theta} & = & -\sin\theta \dfrac{\partial u}{\partial x} & + & \cos\theta \dfrac{\partial u}{\partial y} \end{cases}$$

Solve resulting Linear System for $\dfrac{\partial u}{\partial x}$ & $\dfrac{\partial u}{\partial y}$

$$\frac{\partial u}{\partial y} = \sin\theta \frac{\partial u}{\partial r} + \frac{\cos\theta}{r}\frac{\partial u}{\partial \theta}$$

$$\begin{cases} \cos\theta \dfrac{\partial u}{\partial r} = \cos^2\theta \dfrac{\partial u}{\partial x} + \cos\theta\sin\theta \dfrac{\partial u}{\partial y} \\ \dfrac{-\sin\theta}{r}\dfrac{\partial u}{\partial \theta} = \sin^2\theta \dfrac{\partial u}{\partial x} + -\sin\theta\cos\theta \dfrac{\partial u}{\partial y} \end{cases}$$

Solve resulting Linear System for $\dfrac{\partial u}{\partial x}$ & $\dfrac{\partial u}{\partial y}$

$$\frac{\partial u}{\partial y} = \sin\theta \frac{\partial u}{\partial r} + \frac{\cos\theta}{r}\frac{\partial u}{\partial \theta}$$

$$\begin{cases} \cos\theta \dfrac{\partial u}{\partial r} = \cos^2\theta \dfrac{\partial u}{\partial x} + \cos\theta\sin\theta \dfrac{\partial u}{\partial y} \\ \dfrac{-\sin\theta}{r}\dfrac{\partial u}{\partial \theta} = \sin^2\theta \dfrac{\partial u}{\partial x} + -\sin\theta\cos\theta \dfrac{\partial u}{\partial y} \end{cases}$$

Add the two equations:

$$\implies \cos\theta \frac{\partial u}{\partial r} - \frac{\sin\theta}{r}\frac{\partial u}{\partial \theta} = \left(\sin^2\theta + \cos^2\theta\right)\frac{\partial u}{\partial x} = \frac{\partial u}{\partial x}$$

$$\therefore \quad \frac{\partial u}{\partial x} = \cos\theta \frac{\partial u}{\partial r} - \frac{\sin\theta}{r}\frac{\partial u}{\partial \theta}$$

Use Polar Forms of $\frac{\partial u}{\partial x}$ & $\frac{\partial u}{\partial y}$ for $\frac{\partial^2 u}{\partial x^2}$ & $\frac{\partial^2 u}{\partial y^2}$

$$\frac{\partial u}{\partial x} = \cos\theta \frac{\partial u}{\partial r} - \frac{\sin\theta}{r}\frac{\partial u}{\partial \theta} \qquad \frac{\partial u}{\partial y} = \sin\theta \frac{\partial u}{\partial r} + \frac{\cos\theta}{r}\frac{\partial u}{\partial \theta}$$

$$\begin{aligned}
\frac{\partial^2 u}{\partial x^2} &= \frac{\partial}{\partial x}\left[\frac{\partial u}{\partial x}\right] \\
&= \cos\theta \frac{\partial}{\partial r}\left[\frac{\partial u}{\partial x}\right] - \frac{\sin\theta}{r}\frac{\partial}{\partial \theta}\left[\frac{\partial u}{\partial x}\right] \\
&= \cos\theta \frac{\partial}{\partial r}\left[\cos\theta \frac{\partial u}{\partial r} - \frac{\sin\theta}{r}\frac{\partial u}{\partial \theta}\right] - \frac{\sin\theta}{r}\frac{\partial}{\partial \theta}\left[\cos\theta \frac{\partial u}{\partial r} - \frac{\sin\theta}{r}\frac{\partial u}{\partial \theta}\right] \\
&= \cos\theta\left(\cos\theta \frac{\partial}{\partial r}\left[\frac{\partial u}{\partial r}\right] + \frac{2\sin\theta}{r^2}\frac{\partial u}{\partial \theta} - \frac{\sin\theta}{r}\frac{\partial}{\partial r}\left[\frac{\partial u}{\partial \theta}\right]\right) \\
&\quad - \frac{\sin\theta}{r}\left(-\sin\theta\frac{\partial u}{\partial r} + \cos\theta\frac{\partial}{\partial \theta}\left[\frac{\partial u}{\partial r}\right] - \frac{\cos\theta}{r}\frac{\partial u}{\partial \theta} - \frac{\sin\theta}{r}\frac{\partial}{\partial \theta}\left[\frac{\partial u}{\partial \theta}\right]\right) \\
&= \cos^2\theta\frac{\partial^2 u}{\partial r^2} + \frac{2\cos\theta\sin\theta}{r^2}\frac{\partial u}{\partial \theta} - \frac{\cos\theta\sin\theta}{r}\frac{\partial^2 u}{\partial r\partial \theta} \\
&\quad + \frac{\sin^2\theta}{r}\frac{\partial u}{\partial r} - \frac{\cos\theta\sin\theta}{r}\frac{\partial^2 u}{\partial \theta\partial r} + \frac{\cos\theta\sin\theta}{r^2}\frac{\partial u}{\partial \theta} + \frac{\sin^2\theta}{r^2}\frac{\partial^2 u}{\partial \theta^2} \\
&= \cos^2\theta\frac{\partial^2 u}{\partial r^2} - \frac{2\cos\theta\sin\theta}{r}\frac{\partial^2 u}{\partial r\partial \theta} + \frac{\sin^2\theta}{r^2}\frac{\partial^2 u}{\partial \theta^2} + \frac{\sin^2\theta}{r}\frac{\partial u}{\partial r} + \frac{3\cos\theta\sin\theta}{r^2}\frac{\partial u}{\partial \theta}
\end{aligned}$$

$$\therefore\ \frac{\partial^2 u}{\partial x^2} = \cos^2\theta\frac{\partial^2 u}{\partial r^2} - \frac{2\cos\theta\sin\theta}{r}\frac{\partial^2 u}{\partial r\partial \theta} + \frac{\sin^2\theta}{r^2}\frac{\partial^2 u}{\partial \theta^2} + \frac{\sin^2\theta}{r}\frac{\partial u}{\partial r} + \frac{3\cos\theta\sin\theta}{r^2}\frac{\partial u}{\partial \theta}$$

Use Polar Forms of $\frac{\partial u}{\partial x}$ & $\frac{\partial u}{\partial y}$ for $\frac{\partial^2 u}{\partial x^2}$ & $\frac{\partial^2 u}{\partial y^2}$

$$\frac{\partial u}{\partial x} = \cos\theta \frac{\partial u}{\partial r} - \frac{\sin\theta}{r}\frac{\partial u}{\partial \theta} \qquad \frac{\partial u}{\partial y} = \sin\theta \frac{\partial u}{\partial r} + \frac{\cos\theta}{r}\frac{\partial u}{\partial \theta}$$

$$\begin{aligned}
\frac{\partial^2 u}{\partial y^2} &= \frac{\partial}{\partial y}\left[\frac{\partial u}{\partial y}\right] \\
&= \sin\theta \frac{\partial}{\partial r}\left[\frac{\partial u}{\partial y}\right] + \frac{\cos\theta}{r}\frac{\partial}{\partial \theta}\left[\frac{\partial u}{\partial y}\right] \\
&= \sin\theta \frac{\partial}{\partial r}\left[\sin\theta\frac{\partial u}{\partial r} + \frac{\cos\theta}{r}\frac{\partial u}{\partial \theta}\right] + \frac{\cos\theta}{r}\frac{\partial}{\partial \theta}\left[\sin\theta\frac{\partial u}{\partial r} + \frac{\cos\theta}{r}\frac{\partial u}{\partial \theta}\right] \\
&= \sin\theta\left(\sin\theta\frac{\partial}{\partial r}\left[\frac{\partial u}{\partial r}\right] - \frac{2\cos\theta}{r^2}\frac{\partial u}{\partial \theta} + \frac{\cos\theta}{r}\frac{\partial}{\partial r}\left[\frac{\partial u}{\partial \theta}\right]\right) \\
&\quad + \frac{\cos\theta}{r}\left(\cos\theta\frac{\partial u}{\partial r} + \sin\theta\frac{\partial}{\partial \theta}\left[\frac{\partial u}{\partial r}\right] - \frac{\sin\theta}{r}\frac{\partial u}{\partial \theta} + \frac{\cos\theta}{r}\frac{\partial}{\partial \theta}\left[\frac{\partial u}{\partial \theta}\right]\right) \\
&= \sin^2\theta\frac{\partial^2 u}{\partial r^2} - \frac{2\cos\theta\sin\theta}{r^2}\frac{\partial u}{\partial \theta} + \frac{\cos\theta\sin\theta}{r}\frac{\partial^2 u}{\partial r\partial \theta} \\
&\quad + \frac{\cos^2\theta}{r}\frac{\partial u}{\partial r} + \frac{\cos\theta\sin\theta}{r}\frac{\partial^2 u}{\partial \theta\partial r} - \frac{\cos\theta\sin\theta}{r^2}\frac{\partial u}{\partial \theta} + \frac{\cos^2\theta}{r^2}\frac{\partial^2 u}{\partial \theta^2} \\
&= \sin^2\theta\frac{\partial^2 u}{\partial r^2} + \frac{2\cos\theta\sin\theta}{r}\frac{\partial^2 u}{\partial r\partial \theta} + \frac{\cos^2\theta}{r^2}\frac{\partial^2 u}{\partial \theta^2} + \frac{\cos^2\theta}{r}\frac{\partial u}{\partial r} - \frac{3\cos\theta\sin\theta}{r^2}\frac{\partial u}{\partial \theta}
\end{aligned}$$

$$\therefore\ \frac{\partial^2 u}{\partial y^2} = \sin^2\theta\frac{\partial^2 u}{\partial r^2} + \frac{2\cos\theta\sin\theta}{r}\frac{\partial^2 u}{\partial r\partial \theta} + \frac{\cos^2\theta}{r^2}\frac{\partial^2 u}{\partial \theta^2} + \frac{\cos^2\theta}{r}\frac{\partial u}{\partial r} - \frac{3\cos\theta\sin\theta}{r^2}\frac{\partial u}{\partial \theta}$$

Use Polar Forms of $\frac{\partial^2 u}{\partial x^2}$ & $\frac{\partial u^2}{\partial y^2}$ for $\nabla^2 u$

$$\frac{\partial u}{\partial x} = \cos\theta \frac{\partial u}{\partial r} - \frac{\sin\theta}{r}\frac{\partial u}{\partial \theta} \qquad \frac{\partial u}{\partial y} = \sin\theta \frac{\partial u}{\partial r} + \frac{\cos\theta}{r}\frac{\partial u}{\partial \theta}$$

$$\frac{\partial^2 u}{\partial x^2} = \cos^2\theta \frac{\partial^2 u}{\partial r^2} - \frac{2\cos\theta\sin\theta}{r}\frac{\partial^2 u}{\partial r \partial \theta} + \frac{\sin^2\theta}{r^2}\frac{\partial^2 u}{\partial \theta^2} + \frac{\sin^2\theta}{r}\frac{\partial u}{\partial r} + \frac{3\cos\theta\sin\theta}{r^2}\frac{\partial u}{\partial \theta}$$

$$\frac{\partial^2 u}{\partial y^2} = \sin^2\theta \frac{\partial^2 u}{\partial r^2} + \frac{2\cos\theta\sin\theta}{r}\frac{\partial^2 u}{\partial r \partial \theta} + \frac{\cos^2\theta}{r^2}\frac{\partial^2 u}{\partial \theta^2} + \frac{\cos^2\theta}{r}\frac{\partial u}{\partial r} - \frac{3\cos\theta\sin\theta}{r^2}\frac{\partial u}{\partial \theta}$$

$$\nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \left(\cos^2\theta + \sin^2\theta\right)\frac{\partial^2 u}{\partial r^2} + \frac{\cos^2\theta + \sin^2\theta}{r^2}\frac{\partial^2 u}{\partial \theta^2} + \frac{\cos^2\theta + \sin^2\theta}{r}\frac{\partial u}{\partial r}$$

$$\therefore \quad \boxed{\nabla^2 u = \frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}}$$

Fin.