Gauss Quadrature: Error Term

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EVEN & ODD FUNCTIONS:

f is **odd function** $\iff f(-x) = -f(x)$. g is **even function** $\iff g(-x) = g(x)$.

f is **odd function** $\Rightarrow \displaystyle\int_{-a}^{a} f(x)\,dx = 0$. g is **even function** $\Rightarrow \displaystyle\int_{-a}^{a} g(x)\,dx = 2\int_{0}^{a} g(x)\,dx$.

GAUSS QUADRATURE:

An n-point Gauss Quadrature rule satisfies $\displaystyle\int_a^b \varphi(x)f(x)\,dx \approx \sum_{i=1}^{n} w_i f(x_i)$

where $\varphi(x)$ is a given **weight function**, the x_i's are the **nodes**, the w_i's are the **weights**. Each $x_i \in (a, b)$.

An n-point Gauss Quadrature rule integrates polynomials of degree $(2n-1)$ or less **exactly**.

Unlike Newton-Cotes Quadrature rules, the nodes x_i in Gauss Quadrature rules are **not equidistant**.

When applicable, take advantage of any **symmetry** in $\varphi(x)f(x)$ to simplify the computations.

QUADRATURE ERROR:

An n-point **quadrature rule** $Q^{(n)}(f)$ approximates a **definite integral** $I(f)$ of $f(x)$:

$I(f) \approx Q^{(n)}(f) = \displaystyle\sum_{i=1}^{n} w_i f(x_i)$, where the x_i's are **nodes** and the w_i's are **weights**.

The **error** associated with quadrature rule $Q^{(n)}(f)$ is denoted by $E^{(n)}(f)$: $I(f) = Q^{(n)}(f) + E^{(n)}(f)$

EXAMPLE: Determine the error term of this quadrature rule : $\int_{-1}^{1} f(x)\,dx \approx 2f(0)$

This is a one-point Gauss quadrature rule of the form : $I(f) = Q^{(1)}(f) + E^{(1)}(f)$

An n-point Gauss Quadrature rule integrates polynomials of degree $(2n-1)$ or less **exactly**.

Here, $n = 1$, so this rule exactly integrates **linear polynomials**.

Thus, since **quadratics** are the lowest-degree polynomials that have an **error** with this quadrature, and quadratics have a **constant 2nd derivative**, let $f(x) = x^2$ and $E^{(1)}(f) = kf''(\xi)$, where $\xi \in [-1,1]$

Now, $f(x) = x^2 \Rightarrow f''(x) = 2$. The constant k must be determined :

$$\int_{-1}^{1} x^2\,dx = 2f(0) + E^{(1)}(f) \Rightarrow \frac{2}{3} = 2(0)^2 + kf''(\xi) \Rightarrow kf''(\xi) = \frac{2}{3} \Rightarrow 2k = \frac{2}{3} \Rightarrow k = \frac{1}{3}$$

Hence, the error term of Gauss quadrature rule is : $\boxed{E^{(1)}(f) = \frac{1}{3}f''(\xi)}$, where $\xi \in [-1,1]$ and $f \in C^2[-1,1]$

EXAMPLE: Determine the error term of this quadrature rule :

$$\int_{-1}^{1} x^{10} f(x)\,dx \approx \frac{1}{13} f\left(-\sqrt{\frac{3}{5}}\right) + \frac{4}{143} f(0) + \frac{1}{13} f\left(\sqrt{\frac{3}{5}}\right)$$

This is a 3-point Gauss quadrature rule of the form : $I(f) = Q^{(3)}(f) + E^{(3)}(f)$

An n-point Gauss Quadrature rule integrates polynomials of degree $(2n-1)$ or less **exactly**.

Here, $n = 3$, so this rule exactly integrates **5th-degree polynomials**.

Thus, since **6th-degree polynomials** are the lowest-degree polynomials that have an **error** with this quadrature, and they have a **constant 6th derivative**, let $f(x) = x^6$ and $E^{(3)}(f) = kf^{(6)}(\xi)$, where $\xi \in [-1,1]$

Now, $f(x) = x^6 \Rightarrow f^{(6)}(x) = 6! = 720$. The constant k must be determined :

$$\int_{-1}^{1} x^{10} x^6\,dx = \frac{1}{13} f\left(-\sqrt{\frac{3}{5}}\right) + \frac{4}{143} f(0) + \frac{1}{13} f\left(\sqrt{\frac{3}{5}}\right) + E^{(3)}(f)$$

$$\Rightarrow \int_{-1}^{1} x^{16}\,dx = \frac{1}{13} f\left(-\sqrt{\frac{3}{5}}\right) + \frac{4}{143} f(0) + \frac{1}{13} f\left(\sqrt{\frac{3}{5}}\right) + E^{(3)}(f)$$

$$\Rightarrow \left[\frac{x^{17}}{17}\right]_{-1}^{1} = \frac{1}{13}\left(-\sqrt{\frac{3}{5}}\right)^6 + \frac{4}{143}(0)^6 + \frac{1}{13}\left(\sqrt{\frac{3}{5}}\right)^6 + E^{(3)}(f)$$

$$\Rightarrow \frac{2}{17} = \left(\frac{1}{13}\right)\left(\frac{27}{125}\right) + 0 + \left(\frac{1}{13}\right)\left(\frac{27}{125}\right) + kf^{(6)}(\xi)$$

$$\Rightarrow kf^{(6)}(\xi) = \frac{2332}{27625} \Rightarrow 720k = \frac{2332}{27625} \Rightarrow k = \frac{94}{801741}$$

Hence, the error term of this rule is : $\boxed{E^{(3)}(f) = \frac{94}{801741} f^{(6)}(\xi)}$ where $\xi \in [-1,1]$ and $f \in C^6[-1,1]$

References

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