Newton-Cotes Quadrature: Open & Closed Rules

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KEY CONCEPTS & DEFINITIONS:

\( f \text{ is odd function } \iff f(-x) = -f(x) \) \quad \( g \text{ is even function } \iff g(-x) = g(x) \).

\( f \text{ is odd function } \Rightarrow \int_{-a}^{a} f(x) \, dx = 0 \).
\( g \text{ is even function } \Rightarrow \int_{-a}^{a} g(x) \, dx = 2 \int_{0}^{a} g(x) \, dx \).

CLOSED NEWTON-COTES QUADRATURE RULES:

The \((n+1)\)-point \textbf{closed} Newton-Cotes Quadrature rule satisfies
\[
\int_{a}^{b} \phi(x) f(x) \, dx \approx \sum_{i=0}^{n} w_{i} f(x_{i})
\]
where \( \phi(x) \) is a given weight function, the \( x_{i} \)'s are the nodes, the \( w_{i} \)'s are the weights. Each \( x_{i} \in [a, b] \).

\textbf{NOTE:} The nodes \( x_{i} \)'s are equidistant: \( x_{0} = a, \; x_{i} = x_{0} + ih, \; x_{n} = b \), where \( h = \frac{b-a}{n} \).
The “closed” qualifier means the boundaries of the interval \([a, b] \) are included as nodes.

OPEN NEWTON-COTES QUADRATURE RULES:

The \((n-1)\)-point \textbf{open} Newton-Cotes Quadrature rule satisfies
\[
\int_{a}^{b} \phi(x) f(x) \, dx \approx \sum_{i=1}^{n-1} w_{i} f(x_{i})
\]
where \( \phi(x) \) is a given weight function, the \( x_{i} \)'s are the nodes, the \( w_{i} \)'s are the weights. Each \( x_{i} \in (a, b) \).

\textbf{NOTE:} The nodes \( x_{i} \)'s are equidistant: \( x_{0} = a, \; x_{i} = x_{0} + ih, \; x_{n} = b \), where \( h = \frac{b-a}{n} \).
The “open” qualifier means the boundaries of the interval \((a, b) \) are NOT included as nodes.

REMARKS ABOUT QUADRATURE:

An \( m \)-point Newton-Cotes rule exactly integrates
\[
\left\{ \begin{array}{ll}
\text{polynomials of degree } m \text{ or less,} & \text{if } m \text{ is odd} \\
\text{polynomials of degree } (m-1) \text{ or less,} & \text{if } m \text{ is even}
\end{array} \right.
\]
Quadrature \textbf{tables} provide the nodes & weights for special intervals like \([-1, 1], \; (-1, 1), \; (-\infty, \infty), \; [0, \infty) \).
To convert a quadrature rule from \([-1, 1] \) to \([a, b] \), perform \textbf{substitution}: \( T(x) = Ax + B, T(-1) = a, T(1) = b \).

When applicable, take advantage of any \textbf{symmetry} in \( \phi(x)f(x) \) to simplify the computations.
EXAMPLE: Find the 3-point closed Newton-Cotes quadrature rule satisfying
\[ \int_{-1}^{1} f(x) \, dx \approx w_0 f(x_0) + w_1 f(x_1) + w_2 f(x_2) \]

STEP 1: Determine the nodes \( x_0, x_1, x_2 \)

Since the quadrature is closed, \( n + 1 = 3 \Rightarrow n = 2 \)
\[ h = \frac{b - a}{n} = \frac{1 - (-1)}{2} = 1 \]
\[ x_0 = a = -1, \quad x_1 = x_0 + h = -1 + 1 = 0, \quad x_2 = x_0 + 2h = -1 + 2(1) = 1 = b \]

STEP 2: Build a 3x3 linear system and solve for the weights \( w_0, w_1, w_2 \)

3-point Gauss quadrature rules integrate quadratic polynomials = \( \text{span}\{1, x, x^2\} \) exactly:

Let \( f(x) = 1 \), then \( w_0(1) + w_1(1) + w_2(1) = \int_{-1}^{1} 1 \, dx = 2 \)

Let \( f(x) = x \), then \( w_0(-1) + w_1(0) + w_2(1) = \int_{-1}^{1} x \, dx = 0 \)

Let \( f(x) = x^2 \), then \( w_0(1) + w_1(0) + w_2(1) = \int_{-1}^{1} x^2 \, dx = \frac{2}{3} \)

Simplifying yields the following linear system:

\[
\begin{align*}
w_0 + w_1 + w_2 &= 2 \\
-w_0 + 0 - w_2 &= 0 \\
w_0 + 0 + w_2 &= \frac{2}{3}
\end{align*}
\]

Solving this system using Gaussian elimination yields:

\[
\begin{bmatrix}
1 & 2 & 2/3 \\
-1 & 0 & 1 \\
0 & 1 & 2
\end{bmatrix}
\begin{bmatrix}
w_0 \\
w_1 \\
w_2
\end{bmatrix}
= \begin{bmatrix}
1/3 \\
4/3 \\
1/3
\end{bmatrix}
\]

Hence, the 3-point closed Newton-Cotes quadrature rule is
\[ \int_{-1}^{1} f(x) \, dx \approx \frac{1}{3} f(-1) + \frac{4}{3} f(0) + \frac{1}{3} f(1) \]
EXAMPLE: Find the 3-point open Newton-Cotes quadrature rule satisfying
\[ \int_{-1}^{1} f(x) \, dx \approx w_1 f(x_1) + w_2 f(x_2) + w_3 f(x_3) \]

STEP 1: Determine the nodes \( x_1, x_2, x_3 \)

Since the quadrature is open, \( n - 1 = 3 \Rightarrow n = 4 \)
\[ h = \frac{b - a}{n} = \frac{1 - (-1)}{4} = \frac{1}{2} \]
\[ x_0 = a = -1 \]
\[ x_1 = x_0 + h = -1 + \frac{1}{2} = -\frac{1}{2}, \quad x_2 = x_0 + 2h = -1 + 2 \left( \frac{1}{2} \right) = 0, \quad x_3 = x_0 + 3h = -1 + 3 \left( \frac{1}{2} \right) = \frac{1}{2} \]
\[ x_4 = x_0 + 4h = -1 + 4 \left( \frac{1}{2} \right) = 1 = b \]

STEP 2: Build a 3x3 linear system and solve for the weights \( w_1, w_2, w_3 \)

3-point Gauss quadrature rules integrate quadratic polynomials = span\( \{1, x, x^2\} \) exactly:

Let \( f(x) = 1 \), then \( w_1(1) + w_2(1) + w_3(1) = \int_{-1}^{1} 1 \, dx = 2 \)

Let \( f(x) = x \), then \( w_1 \left( -\frac{1}{2} \right) + w_2(0) + w_3 \left( \frac{1}{2} \right) = \int_{-1}^{1} x \, dx = 0 \)

Let \( f(x) = x^2 \), then \( w_1 \left( \frac{1}{4} \right) + w_2(0) + w_3 \left( \frac{1}{4} \right) = \int_{-1}^{1} x^2 \, dx = \frac{2}{3} \)

Simplifying yields the following linear system:
\[
\begin{align*}
  w_1 + w_2 + w_3 &= 2 \\
  w_1 + 0 + w_3 &= 0 \\
  w_1 + 0 + w_3 &= \frac{8}{3}
\end{align*}
\]

Solving this system using Gaussian elimination yields:
\[
\begin{bmatrix}
  1 & 1 & 1 \\
  1 & 0 & 1 \\
  3 & 0 & 3
\end{bmatrix}
\begin{bmatrix}
  w_1 \\
  w_2 \\
  w_3
\end{bmatrix}
= \begin{bmatrix}
  2 \\
  0 \\
  8
\end{bmatrix}
\Rightarrow
\begin{bmatrix}
  1 & 1 & 1 \\
  0 & 1 & 2 \\
  0 & 0 & 6
\end{bmatrix}
\begin{bmatrix}
  w_1 \\
  w_2 \\
  w_3
\end{bmatrix}
= \begin{bmatrix}
  2 \\
  2 \\
  8
\end{bmatrix}
\Rightarrow
\begin{bmatrix}
  w_1 \\
  w_2 \\
  w_3
\end{bmatrix}
= \begin{bmatrix}
  \frac{4}{3} \\
  \frac{4}{3} \\
  \frac{4}{3}
\end{bmatrix}
\]

Hence, the 3-point open Newton-Cotes quadrature rule is
\[ \int_{-1}^{1} f(x) \, dx \approx \frac{4}{3} f \left( -\frac{1}{2} \right) - \frac{2}{3} f(0) + \frac{4}{3} f \left( \frac{1}{2} \right) \]

Notice that the middle weight, \( w_2 \), is negative. This means some functions with positive area in \([-1, 1]\) have negative quadrature – a contradiction:

For example, take \( f(x) = 1 - \frac{13}{3} x^2 + \frac{10}{3} x^4 \). Then,
\[ \int_{-1}^{1} f(x) \, dx = \int_{-1}^{1} \left( 1 - \frac{13}{3} x^2 + \frac{10}{3} x^4 \right) \, dx = \frac{4}{9} \]
but,
\[ \frac{4}{3} f \left( -\frac{1}{2} \right) - \frac{2}{3} f(0) + \frac{4}{3} f \left( \frac{1}{2} \right) = \frac{4}{3} \left( \frac{1}{8} \right) - \frac{2}{3}(1) + \frac{4}{3} \left( \frac{1}{8} \right) = -\frac{1}{3} \]

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References
