Solving IVP’s : Stability of Runge-Kutta Methods

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NOTATION: $h$ \equiv \text{step size} \quad x_n \equiv x(t) \quad t_{n+1} \equiv t + h \quad x_{n+1} \equiv x(t_{n+1}) \equiv x(t+h)$

Vertical strip $\forall S[t_0, t_a] := [t_0, t_a] \times \mathbb{R}$ \equiv \{(t, x) : t \in [t_0, t_a] \text{ and } x \in \mathbb{R}\}$

Convex Domain \equiv \text{convex connected set} \quad \mathbb{R} \equiv \{\text{real numbers}\}$

SCALAR IVP ASSUMED THROUGHOUT: $\begin{cases} \dot{x}(t) = f(t, x) \\ x(t_0) = x_0 \end{cases}$

TEST IVP: $\begin{cases} \dot{x} = \lambda x \\ x(0) = 1 \end{cases} \implies x = e^{\lambda t}$

ONE-STEP METHODS: One-step methods are either Taylor methods or Runge-Kutta methods. Here, we only consider Runge-Kutta methods as Taylor Methods require taking derivatives.

RUNGE-KUTTA METHOD: $x_{n+1} = x_n + h \varphi(t_n, x_n; h)$, where $\varphi(t, x; h)$ is the kernel of the method.

- A one-step method is convergent \iff $\lim_{h \to 0} \varphi(t, x; h) = f(t, x)$
- A one-step method is consistent \iff $\lim_{h \to 0} \varphi(t, x; h) = f(t, x)$
- A one-step method is stable \iff small changes in IC’s produce small changes in approximate solutions
  \iff $\varphi \in C(\forall S[t, t_a]) \cap \text{Lip}^{(M)}(\forall S[t, t_a]; x)$
- Unless step size $h$ is extremely small ($|h| \ll 1$), i.e. excessively many steps taken, round-off error is no concern.
- Local Truncation Error (LTE) $T_{n+1} := \frac{x(t_{n+1}) - x(t_n)}{h} - \varphi(t_n, x(t_n); h)$ is the error in truncating Taylor series
- A one-step method has order of accuracy $p$ \iff $|T_{n+1}| \leq Ch^p$ for some constant $C > 0$

- 1-variable Taylor Series (useful for LTE): $x(t+h) = \sum_{j=0}^{\infty} \left[ \frac{1}{j!} \left( \frac{d}{dt} \right)^j x(t) \right]$
- 2-variable Taylor Series (useful for LTE): $f(t+h, x+k) = \sum_{j=0}^{\infty} \left[ \frac{1}{j!} \left( \frac{\partial}{\partial t} + k \frac{\partial}{\partial x} \right)^j f(t, x) \right]$
- Global Error $E_n := x_n - x(t_n)$ measures the total error in approximate solution after $n$ steps of method
- A one-step method is convergent \iff $\lim_{h \to 0} |E_n| = 0$ \iff one-step method is consistent & stable
- A one-step method has convergence of order $p$ \iff $|E_n| \leq Kh^p$ for some constant $K > 0$
- Lin-Exp Trump (useful for Global Error): $1 + t \leq e^t \ \forall t \in \mathbb{R}$ and $0 \leq (1+t)^m \leq e^{mt} \ \forall t \geq -1 \ \forall m \geq 0$
- Finite Geometric Series (useful for Global Error): $a + ar + ar^2 + ar^3 + \cdots + ar^N = \sum_{k=0}^{N} ar^k = a \frac{1-r^{N+1}}{1-r}$
- Triangle Inequalities (Stability,LTE,Global Error): $|x| - |y| \leq |x - y| \leq |x - y|$ and $|x + y| \leq |x| + |y|$
EXAMPLE: Consider the following Runge-Kutta Method applied to IVP \( \{ \begin{align*} x'(t) &= f(t,x) \\ x(t_0) &= x_0 \end{align*} \): 

\[
\begin{array}{c|c}
0 & \frac{3}{4} \\
\frac{3}{4} & \frac{3}{4} \\
\frac{8}{3} & \frac{2}{3} \\
\hline
\end{array}
\begin{align*}
F_1 &= hf(t_n, x_n) \\
F_2 &= hf(t_n + \frac{3}{4}h, x_n + \frac{3}{4}F_1) \\
x_{n+1} &= x_n + \frac{2}{3}F_1 + \frac{1}{3}F_2 \\
\end{align*}
\]

\[\iff x_{n+1} = x_n + \frac{8}{3}hf(t_n, x_n) + \frac{2}{3}hf(t_n + \frac{3}{4}h, x_n + \frac{3}{4}hf(t_n, x_n)) \]

Is this RK-Method trustworthy? (that is, is it a convergent method?)

\[x_{n+1} = x_n + h\varphi(t_n, x_n; h) \implies \varphi(t, x; h) = \frac{8}{3}f(t, x) + \frac{2}{3}f(t + \frac{3}{4}h, x + \frac{3}{4}hf(t, x)) \]

\[\lim_{h \to 0} \varphi(t, x; h) = \lim_{h \to 0} \left[ \frac{8}{3}f(t, x) + \frac{2}{3}f\left(t + \frac{3}{4}h, x + \frac{3}{4}hf(t, x)\right) \right] = \frac{8}{3}f(t, x) + \frac{2}{3}f(t, x) = \frac{10}{3}f(t, x) \neq f(t, x) \]

Therefore, this RK-Method is not consistent \(\implies\) it’s not convergent \(\implies\) it’s not trustworthy.
EXAMPLE: Consider the Explicit Euler Method applied to IVP \( \{ x'(t) = f(t, x) \} \):

\[
\frac{F_1 = hf(t_n, x_n)}{x_{n+1} = x_n + F_1} \iff x_{n+1} = x_n + hf(t_n, x_n)
\]

(a) Is Explicit Euler consistent? (b) Is Explicit Euler stable? (c) Is Explicit Euler convergent? (d) Determine the Local Truncation Error (LTE). (e) Determine the Global Error.

(a) \( x_{n+1} = x_n + hf(t_n, x_n) \equiv x_n + h\varphi(t_n, x_n; h) \implies \varphi(t, x; h) = f(t, x) \)

\[
\lim_{h \to 0} \varphi(t, x; h) = \lim_{h \to 0} f(t, x) = f(t, x) \implies \text{Explicit Euler IS consistent}
\]

(b) Assume \( f \in C(\mathbb{V}[t_0, t_s]) \cap \text{Lip}^{(M)}(\mathbb{V}[t_0, t_s]) \). Then, clearly \( \varphi \in C(\mathbb{V}[t_0, t_s]) \)

\[
|\varphi(t, u; h) - \varphi(t, v; h)| = |f(t, u) - f(t, v)| \leq M|u - v| \implies \varphi \in \text{Lip}^{(M)}(\mathbb{V}[t_0, t_s])
\]

\[
\varphi \in C(\mathbb{V}[t_0, t_s]) \cap \text{Lip}^{(M)}(\mathbb{V}[t_0, t_s]) \implies \text{ Explicit Euler IS stable}
\]

(c) Since Explicit Euler is consistent and stable, \( \text{Explicit Euler IS convergent} \)

(d) LTE of Explicit Euler is \( T_{n+1} = \frac{x(t_{n+1}) - x(t_n)}{h} - \varphi(t_n, x(t_n); h) \)

Now, Taylor expand \( x(t_{n+1}) \):

\[
x(t_{n+1}) = x(t_n + h) = x(t_n) + hx'(t_n) + \frac{h^2}{2!}x''(\xi_n), \text{ where } \xi_n \in [t_n, t_{n+1}]
\]

\[
T_{n+1} = \left[ x'(t_n) + \frac{h}{2}x''(\xi_n) \right] - f(t_n, x(t_n)) = \left[ f(t_n, x(t_n)) + \frac{h}{2}x''(\xi_n) \right] - f(t_n, x(t_n)) = \frac{h}{2}x''(\xi_n)
\]

\[
T_{n+1} = \frac{h}{2}x''(\xi_n) \implies |T_{n+1}| \leq \left| \frac{x''(t)}{2} h \right| h \equiv Ch
\]

\[
\text{LTE } |T_{n+1}| \leq Ch \text{ (i.e. Explicit Euler has 1st-order accuracy)}
\]

(e) Global Error is \( E_n = x_n - x(t_n) \)

\[
E_{n+1} - E_n = [x_{n+1} - x(t_{n+1})] - [x_n - x(t_n)] = [x_{n+1} - x_n] - [x(t_{n+1}) - x(t_n)] = h\varphi(t_n, x_n; h) - [x(t_{n+1}) - x(t_n)]
\]

\[
= h\varphi(t_n, x_n; h) - [x(t_{n+1}) - x(t_n)] = h\varphi(t_n, x_n; h) - h\varphi(t_n, x_n; h) + hT_{n+1} = hT_{n+1}
\]

\[
|E_{n+1}| - |E_n| \leq |E_{n+1} - E_n| \leq h|\varphi(t_n, x_n; h) - \varphi(t_n, x(t_n); h)| + h|T_{n+1}| \leq hM|x_n - x(t_n)| + h|T_{n+1}|
\]

\[
|E_{n+1}| \leq |E_n| + hM|E_n| + h|T_{n+1}| = (1 + hM)|E_n| + h|T_{n+1}|
\]

\[
\text{Now, apply this difference equation recursively until } |E_0| \text{ is reached :}
\]

\[
|E_n| \leq (1 + hM)^n|E_0| + [1 + (1 + hM) + (1 + hM)^2 + \cdots + (1 + hM)^{n-1}]h|T_{n+1}|
\]

\[
|E_n| \leq e^{nhM}|E_0| + \left[ \frac{(1 + hM)^n - 1}{(1 + hM) - 1} \right] (h|T_{n+1}|) \leq e^{(t_n-t_0)M}|E_0| + \left[ \frac{e^{(t_n-t_0)M} - 1}{M} \right] |T_{n+1}|
\]

Now, the first term vanishes since \( E_0 := x_0 - x(t_0) = x_0 - x_0 = 0 \)

\[
|E_n| \leq \left[ \frac{e^{(t_n-t_0)M} - 1}{M} \right] |T_{n+1}| \leq \left[ \frac{e^{(t_n-t_0)M} - 1}{M} \right] \left( \frac{h}{2} \frac{|x''(t)|}{\infty} \right) \equiv Kh \left[ \frac{e^{(t_n-t_0)M} - 1}{2M} \right]
\]

\[
\text{Global Error } |E_n| \leq \frac{Kh}{2M} \left[ e^{(t_n-t_0)M} - 1 \right] \text{ (i.e. Explicit Euler has 1st-order convergence)}
\]
References


