7.3. Operational Properties I (Translation theorems)

7.1, 7.2 : Learn how to transform / inverse some basic functions

7.3, 7.4 : Learn rules about how to transform some combination of functions

Thm 1 : (1st translation thm / s-axis)

If $\mathcal{L}(f(t)) = F(s)$ then $\mathcal{L}(e^{at} f(t)) = F(s-a)$ for any real number a

Proof: $\mathcal{L}(e^{at} f(t)) \overset{def}{=} \int_0^\infty e^{-st} e^{ta} f(t)\,dt \Leftrightarrow$

$\Leftrightarrow \underbrace{\int_0^\infty e^{-(s-a)t} f(t)\,dt}_{F(s-a)}$

Example: $\mathcal{L}(t^2 e^{3t})$ $a=3$, $\mathcal{L}(t^2) = \frac{2!}{s^2} = F(s)$

By Thm 1, $\mathcal{L}(t^2 e^{3t}) = F(s-3) = \frac{2}{(s-3)^2}$

Example: $\mathcal{L}(e^{-t}\sin(2t))$ $a=-1$, $\mathcal{L}(\sin(2t)) = \frac{2}{s^2+2^2} = F(s)$

By Thm 1, $\mathcal{L}(e^{-t}\sin(s+1)) = F(s+1) = \frac{2}{(s+1)^2+4}$

Corollary: (1st translation thm for inverse Laplace)

> If $\mathcal{L}^{-1}(F(s)) = f(t)$ then $\mathcal{L}^{-1}(F(s-a)) = e^{at} f(t)$

Example: Find $\mathcal{L}^{-1}\left(\frac{1}{s^2+2s+5}\right)$

Recall: $\mathcal{L}(\sin(at)) = \frac{a}{s^2+a^2}$

Key point: Realize $\frac{1}{s^2+2s+5} = \frac{1}{(s+1)^2+4} = \left(\frac{1}{2}\right)\frac{2}{(s+1)^2+4}$ [a=2]

So, $\frac{1}{s^2+2s+5} = F(s+1)$ for $F(s) = \frac{2}{s^2+2^2}$

Since, $\mathcal{L}^{-1}\left(\frac{2}{s^2+2^2}\right) = \sin(2t)$, then $\mathcal{L}^{-1}(F(s+1)) = e^{-t}\sin(2t)$

Finally, $\mathcal{L}^{-1}\left(\frac{1}{s^2+2s+5}\right) = \frac{1}{2}e^{-t}\sin(2t)$

Example: $\mathcal{L}^{-1}\left(\frac{2s+5}{s^2+6s+34}\right)$

$\frac{2s+5}{s^2+6s+34} = \frac{2s+5}{(s+3)^2+5^2} = 2\frac{(s+3)}{(s+3)^2+5^2} - \frac{1}{(s+3)^2+5^2} = F_1(s+3) - \frac{1}{5}F_2(s+3)$

$F_1(s) = \frac{s}{s^2+5^2}$, $F_2(s) = \frac{5}{s^2+5^2}$

$\mathcal{L}^{-1}(F_1(s)) = \cos(5t)$, $\mathcal{L}^{-1}(F_2(s)) = \sin(5t)$

So, $\mathcal{L}^{-1}(\ldots) = 2\mathcal{L}^{-1}(F_1(s+3)) - \frac{1}{5}\mathcal{L}^{-1}(F_2(s+3)) = 2e^{-3t}\cos(5t) -$
$- \frac{1}{5}e^{-3t}\sin(5t)$

2^{nd} Translation Thm (t-axis)

Motivation: Consider $f(t) = \begin{cases} e^{-2t} & \text{for } 0 \le t < 1 \\ 0 & \text{for } t > 1 \end{cases}$

Find $\mathcal{L}(f(t)) = \int_0^\infty e^{-st} f(t)\,dt$
$= \int_0^1 e^{-st}e^{-2t}\,dt = \int_0^1 e^{-st-2t}\,dt = \frac{e^{-(s+2)t}}{-(s+2)}\Big|_0^1 =$
$= \frac{e^{-(s+2)} - 1}{-(s+2)}$

Original function $f(t)$ is not continuous at $t=1$, but $F(s)$ is continuous

Heaviside function: $u(t) = \begin{cases} 0 & t < 0 \\ 1 & t \ge 0 \end{cases}$

$$u(t-a) = \begin{cases} 0 & t < a \\ 1 & t \ge a \end{cases}$$

Rewrite $f(t) = e^{-2t} + \underbrace{u(t-1)(-e^{-2t})} = \begin{cases} e^{-2t}, & t < 1 \\ 0, & t \ge 1 \end{cases}$

Thm2 (2^{nd} trans thm): If $\mathcal{L}(f(t)) = F(s)$, then $\Rightarrow$

$\mathcal{L}(u(t-a)f(t-a)) = e^{-as}F(s)$ for any $a > 0$

Thm 2': $\mathcal{L}(u(t-a)\,g(t)) = e^{-as}\mathcal{L}(g(t+a))$

Comparing Thm 2 vs Thm2': $f(t-a) = g(t)$

Example: $f(t) = e^{-2t} - u(t-1)e^{-2t}$

$\mathcal{L}(f(t)) = \mathcal{L}(e^{-2t}) - \mathcal{L}(u(t-1)e^{-2t})$

(By Thm 2') $= \frac{1}{s+2} - e^{-s}\mathcal{L}(e^{-2(t+1)})$

$= \frac{1}{s+2} - e^{-s}\mathcal{L}(e^{-2t})e^{-2}$

$= \frac{1}{s+2} - e^{-(s+2)}\frac{1}{s+2} = \frac{1-e^{-(s+2)}}{s+2}$

Example:

Example: $f(t) = \begin{cases} f_1(t) & , 0 \le t < t_1 \\ f_2(t) & , t_1 \le t < t_2 \\ f_3(t) & , t \ge t_2 \end{cases}$

Rewrite $f(t) = f_1(t) + u(t-t_1)(f_2(t) - f_1(t)) + u(t-t_2)(f_3(t) - f_2(t)) =$

$= f_1 + u(t-t_1)(f_2 - f_1) + u(t-t_2)(f_3 - f_2)$

Corollary: $\mathcal{L}^{-1}(e^{-as}F(s)) = u(t-a)\mathcal{L}^{-1}(F(s))\Big|_{t-a}$

Example: $\mathcal{L}^{-1}\left(\frac{e^{-s}}{s^2+s}\right) \overset{cor}{=} u(t-1)\mathcal{L}^{-1}\left(\frac{1}{s^2+s}\right)\Big|_{t-1}$ $[a=1]$

$\mathcal{L}^{-1}\left(\frac{1}{s^2+s}\right) = \mathcal{L}^{-1}\left(\frac{1}{s} - \frac{1}{s+1}\right) = 1 - e^{-t}$

Finally, $\mathcal{L}^{-1}\left(\frac{e^{-s}}{s^2+s}\right) = u(t-1)\left(1 - e^{-(t-1)}\right)$

Example: Solve $y'' - y = \begin{cases} 0 & \text{for } t < 1 \\ 2 & \text{for } t \ge 1 \end{cases}$, $y(0) = 4$, $y'(0) = 2$

Second order, linear, homogeneous for $t < 1$

Realize: $y'' - y = 2u(t-1)$

$\mathcal{L}(y'' - y) = 2\mathcal{L}(u(t-1)) \Rightarrow s^2 Y - s y(0) - y'(0) - Y = 2\frac{e^{-s}}{s}$

$[\mathcal{L}(1) = \frac{1}{s}]$ $(s^2 - 1)Y - 4s - 2 = 2\frac{e^{-s}}{s}$ + $\frac{4s+2}{s^2-1}$

$y = \mathcal{L}^{-1}(Y) = \mathcal{L}^{-1}\left(\frac{2e^{-s}}{s(s^2-1)}\right) + \mathcal{L}^{-1}\left(\frac{4s+2}{s^2-1}\right)$

$2\mathcal{L}^{-1}\left(\frac{e^{-s}}{s(s^2-1)}\right) \overset{a=1}{=} 2u(t-1)\mathcal{L}^{-1}\left(\frac{1}{s(s^2-1)}\right)\Big|_{t-1}$

Ratial Function: $\frac{1}{s(s^2-1)} = \frac{1}{s(s-1)(s+1)} = \frac{A}{s} + \frac{B}{s-1} + \frac{C}{s+1}$ ⊜

$\begin{cases} A = -1 \\ B = 1/2 \\ C = 1/2 \end{cases} \Rightarrow$ ⊜ $-\frac{1}{s} + \frac{1/2}{s-1} + \frac{1/2}{s+1}$

$\mathcal{L}^{-1}\left(\frac{1}{s(s^2-1)}\right) = \mathcal{L}^{-1}\left(-\frac{1}{s} + \frac{1/2}{-1+s} + \frac{1/2}{s+1}\right)$ ⊜

⊜ $-1 + \frac{1}{2}e^{t} + \frac{1}{2}e^{-t}$

$2\mathcal{L}^{-1}\left(\frac{1}{s(s^2-1)}\right)\Big|_{t-1} = -2 + e^{t-1} + e^{-(t-1)}$

(2) $\mathcal{L}^{-1}\left(\frac{4s+2}{s^2-1}\right) = \mathcal{L}^{-1}\left(\frac{3}{s-1} + \frac{1}{s+1}\right) = 3e^{t} + e^{-t}$

$$\frac{4s+2}{s^2-1} = \frac{D}{s-1} + \frac{E}{s+2} \;\ldots\; D=3,\ E=1$$

Final answer: $y = u(t-1)\left(-2+e^{t-1}+e^{1-t}\right)+3e^{t}+e^{-t}$