September 8th, 2020

- Section 2.3 Linear Equations (1st Order)
- Def: A FOLE (First order linear equation) is an equation in the form: $y' + P(x)y = f(x)$ ⇐ (standard form)
  not separable in general
- Method of solving: LHS is the derivative of some product (product rule) $(UV)' = U'V + UV'$ ⇐ compare w/ LHS
- look for an integrating factor $\mu(x)$

$$\mu(x)y' + \mu(x)P(x)y = \mu(x)f(x)$$

$U' = y'$ $\quad U = y$

$V = \mu(x)$ $\quad V' = \mu(x)P(x)$ $\quad$ need $\mu(x)P(x) = V' = \mu'(x)$

$$\boxed{\int \frac{\mu'(x)}{\mu(x)} = \int P(x)}$$

determines the integrating factor

$$\ln(\mu(x)) = \int P(x)\,dx$$

$$\therefore \mu(x) = e^{\int P(x)dx}$$

- Once $\mu(x)$ is determined, then the equation becomes

$$(\mu(x)y)' = \mu(x)f(x)$$

taking integration of both sides

$$\mu(x)y = \int \mu(x)f(x)$$

$$\therefore y = \frac{\int \mu(x)f(x)\,dx}{\mu(x)}$$

- In summary $y' + P(x)y = f(x)$

$\mu(x) = e^{\int P(x)dx}$: integrating factor

$$\int (\mu(x)y)'\,dx = \int \mu(x)f(x)\,dx$$

$$y = \frac{\int \mu(x)f(x)\,dx}{\mu(x)}$$

- Note: homogeneous F.O.L.E. = $(f(x) \equiv 0)$
  then $y = \frac{c}{\mu(x)}$

- Ex) $xy' + y = \underset{f(x)}{0}$ ← not in standard form, need to manipulate
  $y' + \frac{1}{x}y = 0$ ← in standard form
  $p(x) = \frac{1}{x}$, $\mu(x)\ e^{\int \frac{1}{x} dx} = e^{\ln(x)} = x$
  $p(x) = \frac{1}{x}$, $\mu(x) = x$
  $(yx)' = x \cdot 0$ ⟸ $(\mu(x)y)' = \mu(x)f(x)$
  $yx = \int 0\,dx$
  $yx = c$
  $\boxed{y = \frac{c}{x}}$

- Ex) $y' + \frac{y}{x} = 4x^2$ ⟸ F.O.L.E. in standard form
  $p(x) = \frac{1}{x}$, $\mu(x) = x$
  $(xy)' = \mu(x)f(x) = x(4x^2)$
  $xy = \int 4x^3\,dx$
  $xy = x^4 + c$
  $\boxed{y = x^3 + \frac{c}{x}}$

- Observation: Solution to $(y' + \frac{y}{x} = 0)$ is $\frac{c}{x}$
  $(y' + \frac{y}{x} = 4x^2)$ is $x^3 + \frac{c}{x}$

- Principle: The solution to a non-homogeneous equation is the sum of the two solutions $y = y_c + y_p$ where y_c = complementary solution = solution to the corresponding homogeneous eq. and y_p = particular solution

- Ex) $y' + 2y = \begin{cases} 2 & 0 \le x \le 3 \\ 0 & x > 3 \end{cases}$ IVP $y(0) = 0$

- $0 \le x \le 3$, $y' + 2y = 2$ $y(0) = 0$

$p(x) = 2$, $\mu(x) = e^{\int 2 dx}$

$\mu(x) = e^{2x}$

$(\mu(x) y)' = \mu(x) f(x)$

$\mu y = \int 2e^{2x} dx$

$\mu y = e^{2x} + C$

$y = \frac{e^{2x} + C}{e^{2x}} = 1 + \frac{C}{e^{2x}}$ ⇐ general sol

$y(0) = 0$

$1 + \frac{C}{e^0} = 0$

$1 + C = 0$

$C = -1$

$\therefore y = 1 - \frac{1}{e^{2x}}$

- $x \ge 3$ $y' + 2y = 0$

...math...

$y = \frac{C}{e^{2x}}$

$y(3) = 1 - \frac{1}{e^{2x}}$

$= 1 - \frac{1}{e^6} = \frac{C}{e^6}$

$1 = \frac{C+1}{e^6} \Rightarrow C + 1 = e^6 \Rightarrow C = e^6 - 1$

- Final answer $y = \begin{cases} 1 - \frac{1}{e^{2x}} & 0 \le x \le 3 \\ \frac{e^6 - 1}{e^{2x}} & x > 3 \end{cases}$

- 2.4 Exact Equations
- Theoretical discussion:
  Differential of a function of 2 variables
  $F = F(x,y)$
  $dF = Mdx + Ndy$, $M = \frac{\partial F}{\partial x}$ $N = \frac{\partial F}{\partial y}$
  $\Downarrow$
  $M + N\frac{dy}{dx} = M + Ny'$

- Definition: The expression $M(x,y)dx + N(x,y)dy$ is exact if it corresponds to the differential of some function $F(x,y)$.
  A first order ODE of the form $Mdx + Ndy = 0$ is exact if the LHS is exact

- In Practice: Criteria for being exact
  $Mdx + Ndy$ is exact iff $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$

- Ex) $xdx + ydy = 0 \Rightarrow$ exact
  $m(x,y) = x$
  $N(x,y) = y$
  $\frac{\partial M}{\partial y} = 0$ $\frac{\partial N}{\partial x} = 0$

- Ex) $xydx + (2x^2 + 3y^2 - 20)dy = 0$
  $M = xy$ $N = 2x^2 + 3y^2 - 20$
  $\frac{\partial M}{\partial y} = x \neq \frac{\partial N}{\partial x} = 4x$

- Method of Solving
- Case 1: Exact $mdx + ndy = 0$
  $F = \text{constant}$ $M = \frac{\partial F}{\partial x}$ $N = \frac{\partial F}{\partial y}$
  Need to determine F

$M = \frac{\partial F}{\partial x} \Rightarrow F(x,y) = \int M(x,y)dx + g(y)$

$N = \frac{\partial F}{\partial y} \quad N(x,y) = \frac{\partial}{\partial y}\left(\int M(x,y)dx\right) + g'$

$g' = N - \frac{\partial}{\partial y}\left(\int M(x,y)dx\right)$

Solve for $g(y) \Rightarrow F(x,y)$

ex) $x\,dx + y\,dy = 0$

$M = x \quad N = y$

$\frac{\partial F}{\partial x} = M, \quad F = \int M dx + g(y)$
$= \int x dx + g(y)$
$= \frac{1}{2}x^2 + g(y)$

$\frac{\partial F}{\partial y} = N, \quad y = N = \frac{\partial}{\partial y}\left(\frac{x^2}{2}\right) + g'(y) = g'(y)$

$g'(y) = y \Rightarrow g = \int y dy = y^2/2 + C$

$F = \frac{x^2}{2} + \frac{y^2}{2} + C$

the implicit solution to the ODE is $F = \text{constant}$
$\Leftrightarrow \frac{x^2}{2} + \frac{y^2}{2} = C$