09/15/20 $\boxed{\tilde{M}dx + \tilde{N}dy = 0}$

$\frac{\partial \tilde{M}}{\partial y} \neq \frac{\partial \tilde{N}}{\partial x} \Rightarrow$ not exact

So, we $\times$ by μ — μ-integral factor

$(\mu\tilde{M})dx + (\mu\tilde{N})dy = 0$

$M = \mu\tilde{M} \quad N = \mu\tilde{N}$

To be exact: $\mu_y\tilde{M} + \mu\tilde{M}_y - \mu_x\tilde{N} - \mu\tilde{N}_x = 0 \Rightarrow$ generally not possible

BUT:

Special cases

① $\boxed{\frac{\tilde{M}_y - \tilde{N}_x}{\tilde{N}} = f(x)}$ (not depending on y) $\Rightarrow \boxed{\mu = e^{\int f(x)dx}}$

② $\boxed{\frac{\tilde{M}_y - \tilde{N}_x}{\tilde{M}} = g(y)}$ (not depending on x) $\Rightarrow \boxed{\mu = e^{-\int g(y)dy}}$

Example:

$$xy\,dx + (2x^2 + 3y^2 - 20)\,dy = 0$$

$\tilde{M} = xy \quad \tilde{N} = 2x^2 + 4y^2 - 20$

$\tilde{M}_y = \frac{\partial \tilde{M}}{\partial y} = x \neq \frac{\partial \tilde{N}}{\partial x} = 4x \Rightarrow$ Non.-exact

Observe: $\frac{\tilde{M}_y - \tilde{N}_x}{\tilde{M}} = \frac{x - 4x}{xy} = -\frac{3}{y} = g(y)$

$\mu = e^{\int \frac{3}{y}dy} = e^{3\ln y} = y^3$

$y^3(xy\,dx + (2x^2 + 3y^2 - 20)\,dy = 0)$

$xy^4dx + y^3(2x^2 + 3y^2 - 20)dy = 0 \Rightarrow M = xy^4 \quad N = y^3(2x^2 + 3y^2 - 20)$

Now $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} \Rightarrow$ exact $\Rightarrow$ We can use method for exact eq

$F(x,y) = \int M dx + g(y) = \int xy^4 dx + g(y) = y^4\frac{x^2}{2} + g(y)$

$N = \frac{\partial F}{\partial y} = \frac{d}{dy}\left(\frac{y^4x^2}{2}\right) + g'(y) = 2y^3x^2 + g'(y)$

$g(y) = \int g'(y)\,dy = \frac{y^6}{2} - 5y^4$

general solution

$F(x,y) = \frac{x^2y^4}{2} + \frac{y^6}{2} - 5y^4 \Rightarrow \frac{x^2y^4}{2} + \frac{y^6}{2} - 5y^4 = C$

Homogeneous functions

$\boxed{y' = f(x,y)\ ,\ f(x,y)}$ - homogeneous

Homogeneous functions: $f(tx, ty) = t^{\alpha} f(x,y)$, α - const

$\Longleftrightarrow$

$f(x,y) = F\left(\frac{y}{x}\right)$

Example: $f(x,y) = \frac{y^2}{x^2} + \frac{y}{x} \Rightarrow f(tx, ty) = \frac{(ty)^2}{(tx)^2} + \frac{ty}{tx} = \frac{y^2}{x^2} + \frac{y}{x} =$

$= t^0 f(x,y)$ - homogeneous ($\alpha = 0$)

! Order of x and y equals to each other

$[\ f(x,y) = \frac{y^2}{x^2} + \frac{y}{x} \quad \left(\frac{y}{x} = u\right) \quad f(x,y) = u^2 + u\ ]$

Method of solving:

Sub: $u = y/x$

$\left[\begin{array}{l} u = \frac{y}{x} \Rightarrow y = ux \\ y' = u'x + u \\ \text{Do it to change in the equation} \end{array}\right] \rightarrow$ Use **IF** the equation is homogeneous

$f(x,y) = y'$ - homogeneous $\quad u'x + u = F(u)$

$u'x = F(u) - u$

Do it to have a separable eq $\longrightarrow$ $\frac{du}{F(u) - u} = \frac{dx}{x}$ $\rightarrow$ separable

Example: $y' = \frac{y^2}{x^2} + \frac{y}{x}$ - homogeneous $[\ u = \frac{y}{x},\ y' = u'x + u\ ]$

$u'x + u = u^2 + u$

$u'x = u^2 \Rightarrow \frac{du}{dx} x = u^2 \Rightarrow \frac{du}{u^2} = \frac{dx}{x} \Rightarrow -\frac{1}{u} = \ln x + C$

Change: $u \leftrightarrow y$ $\quad -x = y(\ln|x| + C)$ - implicit solution

$y = -\frac{x}{\ln|x| + C}$ - explicit

Bernoulli's equation

Standard form: $\boxed{y' + P(x)y = f(x)y^n}$

Method of solving: $u = y^{1-n}$

$\left[\begin{array}{l} u = y^{1-n} \\ u' = (1-n)y^{-n} \cdot y' \\ \text{multiply by } (1-n)y^{-n} \\ u' + P(x)y(1-n)y^{-n} = f(x)y^n \cdot y^{-n}(1-n) \\ u' + (1-n)P(x)y^{1-n} = f(x)(1-n) \end{array}\right]$

$u' + (1-n)P(x)u = (1-n)f(x)$

first order linear equation.

Example: $y'+y=y^4$ $\rightarrow u=y^{1-4}=y^{-3}$

$P(x)=1, \quad f(x)=1, \quad n=4$

$u'+\underbrace{(1-4)}_{-3}\underbrace{P(x)}_{1}\cdot u=\underbrace{(1-4)}_{-3}\underbrace{f(x)}_{1}$

$u'-3u=-3 \rightarrow \mu=e^{\int -3dx}=e^{-3x}$

$\left[(\mu u)'=\mu' u+\mu u' = \mu u'+P(x)u\mu \quad \mu=e^{\int P(x)dx} \right]$

$(\mu u)'=\mu(-3)=\int e^{-3x}(-3) \rightarrow \mu u=e^{-3x}+C$

$y^{-3}=u=\frac{e^{-3x}+C}{e^{-3x}}=1+Ce^{3x} \rightarrow y^{-3}=1+Ce^{3x}$ - implicit solution

Standard form: $\boxed{y'=f(Ax+By+C)}$

Sub: $u=Ax+By+C$

$u'=Bf(u)+A$

$\rightarrow$ separable

$\left[\begin{array}{l} u=Ax+By+C \\ u'=A+By' \\ \text{multiply with } B \text{ and add } A \\ u'=Bf(u)+A \end{array} \right]$

Example: $y'=\underbrace{e^{-(x+y)}-1}_{f(x+y)}$ with $y(0)=0$

$u=x+y, \ A=1, \ B=1 \rightarrow u'=(e^{-u}-1)+1$

$u'=e^{-u} \rightarrow$ separable

$\frac{du}{dx}=e^{-u} \rightarrow \frac{du}{e^{-u}}=dx \rightarrow e^{u}=x+C$

$e^{x+y}=x+C \leftarrow$ general implicit solution

$y(0)=0, \ x=0, \ y=0 \mapsto e^{0}=0+C \rightarrow C=e^{0}=1$

$e^{x+y}=x+1$

Review

1. Separable	separate variable $\rightarrow \int$
2. 1^{st} order linear equation	integrating factor $\mu \rightarrow \int$
3. Exact DE Non exact $\rightarrow$ Exact	find $F(x,y)$ and set $F=C \rightarrow \int$ another integrating factor $\mu \rightarrow \int$
4. Homogeneous F	$(u=\frac{y}{x}) \rightarrow$ Separable

(4) Bernoulli's equation	$(u = y^{1-n}) \rightarrow$ First order Linear eq.
(5) $y' = f(Ax+By+C)$	$(u = Ax+By+C) \Rightarrow$ separable eq.