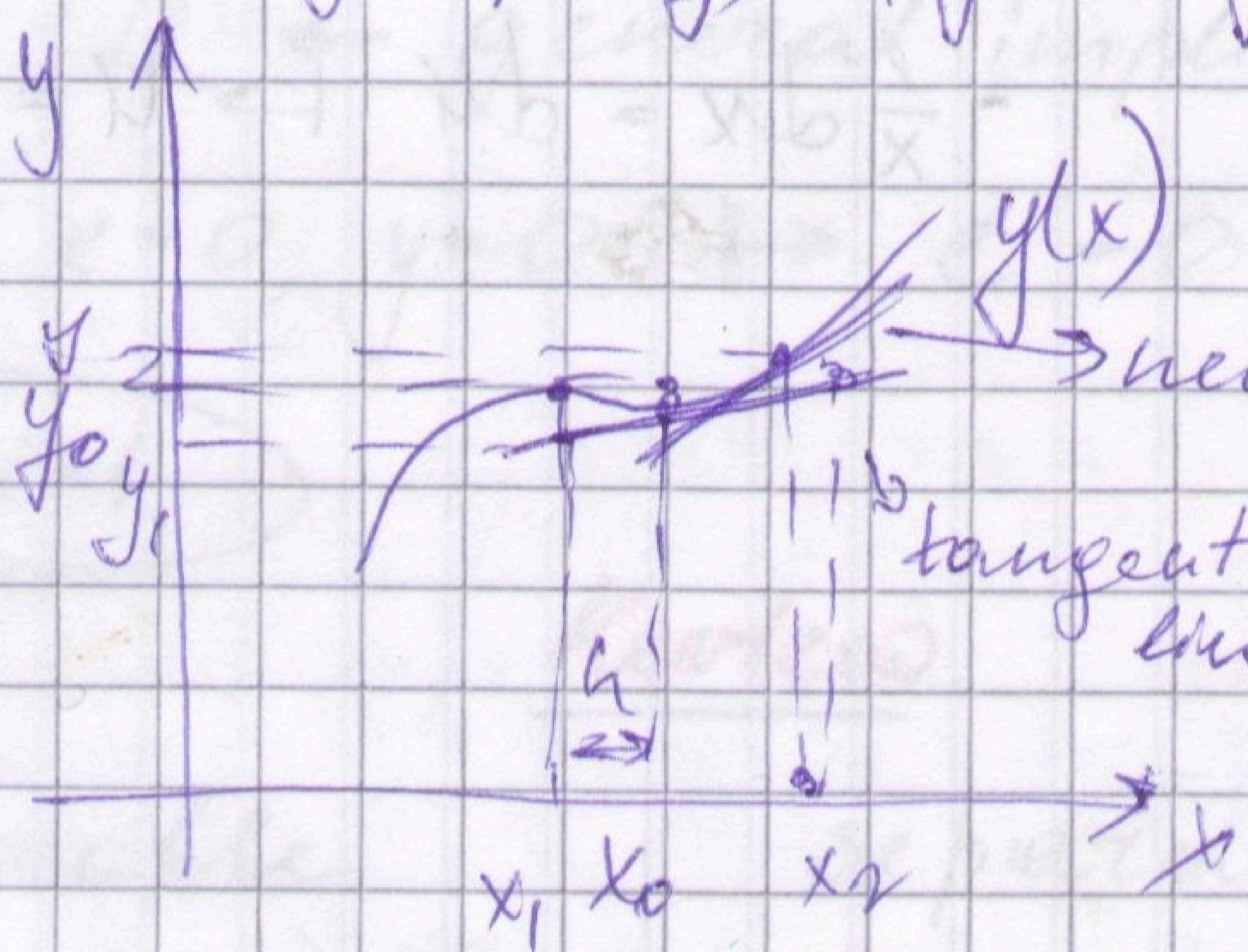
26.

A numerical method

- Analytic solution VS Numerical solution
- Tangent line

Consider $y' = f(x, y)$, $y(x_0) = y_0$



y' = slope of the tangent line

Numerical sol = patching up segment of tangent line

- Euler's method:

Linearization: (tangent line)

$L(x) = f(x_0, y_0)(x - x_0) + y_0$

eq. of the line trough (x_0, y_0) with slope $f(x_0, y_0)$

Specify step size h

x_0 $\quad y_0$

$x_1 = x_0 + h \quad y_1 = f(x_0, y_0)(x_1 - x_0) + y_0$

$x_2 = x_1 + h \quad y_2 = \underbrace{f(x_1, y_1)}_{\text{new slope at } (x_1, y_1)}(x_2 - x_1) + y_1$

Ex: $y' = 2xy$, $y(1) = 1$ $\quad h = .1$

$x_0 = 1 \quad y_0 = 1$

$x_1 = 1 + 0.1 = 1.1 \quad y_1 = (.2 \times 1 \times 1) \ldots$

$x_2 = x_1 + h = 1.2 \quad y_2 = (.2 \times 1.1 \times 1.02) = 1.0424$

Num: $x_1 = 1.1$, $y_1 = 1.02$, $x_2 = 1.2$, $y_2 = 1.0424$

Actual: $x_1 = 1.1$, $y_1 = 1.0212$, $x_2 = 1.2$, $y_2 = 1.0450$

Abs error = |True value − Num. value|

Percentange Relative error $= \dfrac{\text{Abs error}}{|\text{True value}|} \cdot 100$

3.1. Linear Method

- Growth and Decay: Population (biology), economy (economics), Decomposition of Chemical materials (physics, chem)

$\frac{dx}{dt} = kx$, $x(t_0) = x_0$ $\quad k$ - constant

- Newton's law of cooling/warming

$\frac{dT}{dt} = k(T - T_m)$

T: temperature of smth

T_m: ambient temperature

k: constant

Chapter 4

4.1 Theory of Linear Equations

Def: An n(th) order linear ODE is an eq. of the form:

$$a_n(x)y^{(n)} + a_{n-1}(x)y^{(n-1)} + \ldots + a_1(x)y' + a_0(x) = g(x) \quad (1)$$

Remark: - Derivatives of y appear isolated from each other

- Each is multiplied by an expression of x (not involving any y)

IVP: (1) + set of exactly n initial conditions

Remark: - General sol. expectedly has n diff. const

- n initial conditions $\Rightarrow$ n eq. on n const

- Homogeneous linear ODE: (1) but $g(x) \equiv 0$

- Nonhomogeneous linear ODE: (1) but $g(x) \neq 0$

Thm (Existence & Uniqness for Linear IVPs)

If the function $\{a_j(x)\}_{j=0}^{n}$ and $g(x)$ are continuous on an interval $I = \{a \leq x \leq b\}$ and $a_n(x) \neq 0$ for all $x \in I$ then there is a unique sol to IVP

Ex: $(x-2)y'' + 3y = x$

$n = 2$, $a_2(x) = x-2$, $a_1(x) = 0$, $a_0(x) = 3$ $g(x) = x$

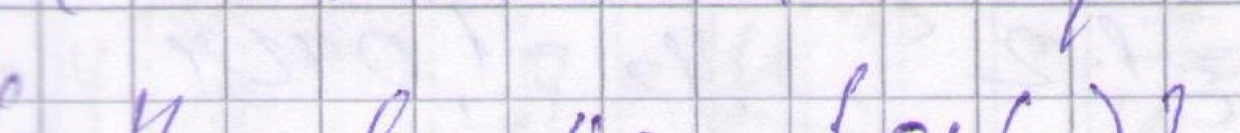
2^{nd} order linear nonhomogeneous ODE

- $a_2(x), a_1(x), a_0(x), g(x)$ are continuous on $\mathbb{R}$

- $a_2(x) = x - 2 = 0$ iff $x = 2$

$\to$ avoid it

Largest interval: $-\infty < x < 2$



$x_0 = 0$ $\quad$ 2

Principal of superposition

$\boxed{\text{Non homog Vs Homogen}}$

If $y_1, y_2, \ldots, y_n$ are functions

$D = \frac{d}{dx}$, $L = a_n(x)D^n + \ldots + a_1(x)D + a_0(x)$

Non-hom: $Ly - g(x) \neq 0$ (2)

Homogen: $Ly = 0$ (3)

If $y_1, \dots, y_n$ are n functions satisfying (3) then $y = C_1 y_1 + \dots + C_n y_n$ is also a solution of (3) for any const ~~$c_1, \dots, c_n$~~ $C_1, \dots, C_n$

Def: A set of function $y_1, \dots, y_n$ are called linearly independent if

$$C_1 y_1(x) + C_2 y_2(x) + \dots + C_n y_n(x) = 0, \quad \forall x \in I$$

iff $C_1 = C_2 = C_3 = \dots = C_n = 0$

Def: A lin. indep. set of exactly n functions $y_1, \dots, y_n$ satisfying (3) is called a fundamental set

Then the general sol to (3) is given by $y = C_1 y_1(x) + \dots + C_n y_n(x)$

(+) How to determine dependency:

Def: Given a set of functions $y_1, \dots, y_k$ we define the Wronskian by

$$W = W(y_1, \dots, y_n) \begin{vmatrix} y_1 & y_2 & \dots & y_k \\ y_1' & y_2' & \dots & y_k' \\ \vdots & & & \vdots \\ y_1^{(k-1)} & y_2^{(k-1)} & \dots & y_k^{(k-1)} \end{vmatrix}$$ determinent

Example: $y_1 = e^{-3x}$, $y_2 = e^{4x}$

$$W(y_1, y_2) = \begin{vmatrix} e^{-3x} & e^{4x} \\ -3e^{-3x} & 4e^{4x} \end{vmatrix} = 4e^{x} + 3e^{x} = 7e^{x} \neq 0 \Rightarrow \text{independent}$$

$\{y_1, y_2\}$ - lin indep

Thm: (Wronskian Test)

The function $y_1, \dots, y_n$ linearly indep iff $W(y_1, \dots, y_n) \neq 0$ for $\forall x \in I$

Thm: There exists a fundamental set for (3) $\Longleftrightarrow$

$\Longleftrightarrow$ There is a general sol of the form $C_1 y_1(x) + \dots + C_n y_n(x)$

(+) For (2) (the non-homog eq)

Thm: Consider $Ly = g(x)$. If y_p is any particular sol and y_c is the complementary sol (sol to $Ly = 0$) then the general solution is $y = y_c + y_p$

Ex: Consider $y'' + y = 1 + x^2$

$y_p = x^2 - 1$, check that $\begin{cases} y_p'' = 2 \\ y_p'' + y = 2 + x^2 - 1 = 1 + x^2 \end{cases}$ $\mapsto$ y_p is particular solution

The corresponding homogeneous eq: $y'' + y = 0$

$\{y_1, y_2\} \to$ is fund set $\Longleftarrow$ $y_c = C_1 \cos x + C_2 \sin x$

$y_1 = \cos x, \quad y_2 = \sin x$

The general solution to the non-hom. eq is $y = x^2 - 1 + \cos x \, C_1 + C_2 \sin x$