Section 4.2 – Reduction of order

- Consider a 2nd order homo. linear ODE:

  1. $y'' + P(x)y' + q(x)y = 0$

  by our theory, general solution: $y = c_1 y_1 + c_2 y_2$ (y_1, y_2 lin. ind.) Suppose that y_1 is known, Find y_2.

- Idea – $y_2(x) = v(x)\, y_1(x)$

  $y_2' = v' y_1 + v y_1'$ (product rule, 1st deriv.)

  $y_2'' = v'' y_1 + 2v' y_1' + v y_1''$

  Sub in (1) yeilds: $\underbrace{v'' y_1 + 2v' y_1' + v y_1''}_{y_2''} + \underbrace{p v' y_1 + p v y_1'}_{p(x) y_2'} + \underbrace{q v y_1}_{q(x) y_2} = 0$

  $\underbrace{v(y_1'' + p y_1' + q y_1)}_{=0} + v'' y_1 + 2v' y_1' + p v' y_1 = 0$

  $v'' y_1 + 2v' y_1' + p v' y_1 = 0$ *

  $w = v'$, $w' = v''$ $\longleftrightarrow$ $w' y_1 + w(2y_1' + p y_1) = 0$ (**) (**) 1st order (reduction) linear eq., to slove 1st order – integrator factor

  $w' + w(2\, y_1'/y_1 + p) = 0$, Integrating Factor: $\mu = e^{\int 2 y_1'/y_1 + p} = y_1^2 e^{\int p(x)dx}$ (***) $\longleftrightarrow$ $(w, \mu)' = 0$ so, $w\mu = c$

  $w = c/\mu = c / y_1^2 e^{\int p(x)dx}$

- $v(x) = \int w = c \int 1/y_1^2 e^{\int p}$
- $y_2(x) = v(x)\, y_1(x)$

  ] very important to remember

- Ex: $y'' - 4y' + 4y = 0$ $y_1 = e^{2x}$ is a solution, find y_2.

  $p(x) = -4$, $y_1^2 e^{\int p} = e^{4x} e^{-4x} = 1$, $v(x) = c\int \frac{1}{1} dx = cx$, $y_2 = v(x) y_1(x) = cxe^{2x} = xe^{2x}$

- Ex: $xy'' + y' = 0$, $y_1 = \ln(x)$, find y_2

  $x \neq 0$, $y'' + \frac{1}{x} y' = 0$, $p(x) = 1/x$, $y_1^2 e^{\int p(x)dx} = (\ln(x))^2 e^{\ln x} = (\ln(x))^2 \cdot x$, so $v(x) = \int \frac{1}{1 \cdot (\ln(x))^2} dx$, $u = \ln(x)$ $du = 1/x\, dx$

  $\int 1/u^2\, du = -1/u = -1/\ln(x)$, $y_2 = v(x) y_1(x) = 1/\ln(x) \cdot \ln(x) = -1$

Section 4.3 – Homo linear Constant Coefficient Eq.

- Second order case: $ay'' + by' + cy = 0$ ①, where a, b, c are constants
- Solutions include factor e^{rx}
- Consider characteristic polynomial – $ar^2 + br + c = 0$ ②
- Explanation – $y = e^{rx}$, $y' = re^{rx}$, $y'' = r^2 e^{rx}$ sub in eq. ①: $ar^2 e^{rx} + bre^{rx} + ce^{rx} = 0$

  $e^{rx}(ar^2 + br + c) = 0$, $ar^2 + br + c = 0$

- Quadratic polynomial: Discriminat: $\Delta = b^2 + ac$
- Case 1: $\Delta > 0$, 2 real roots $r_1 \neq r_2$

  2 independ. sol. $e^{r_1 x}$, $e^{r_2 x}$, general sol: $y = c_1 e^{r_1 x} + c_2 e^{r_2 x}$

- Case 2: $\Delta = 0$, real double root, $r_1 = r_2$

  1 independ. sol. $e^{r_1 x}$, by reduction of order 2nd sol: $xe^{r_1 x}$, general sol: $y = c_1 e^{r_1 x} + c_2 x e^{r_1 x}$

- Case 3: $\Delta < 0$, no real roots, but complex roots, $r = \alpha \pm i\beta$

  2 ind. sol. $e^{\alpha x}\cos(\beta x)$, $e^{\alpha x}\sin(\beta x)$, general sol: $y = c_1 e^{\alpha x}\cos(\beta x) + c_2 \sin(\beta x)$

  explanation: Euler Formula – $e^{i\beta} = \cos\beta + i\sin$ and $e^{-i\beta} = \cos(\beta) - i\sin\beta$, using these formulas give you the inde. sol.

  - Ex: $x^2 + 1 = 0$, $x = \pm i$, $i = \sqrt{-1}$

- Ex: $y'' + y = 0$, characteristic poly: $x^2 + 1 = 0$, $x = \pm i$ so $r = \pm i$, $\alpha = 0$, $\beta = 1$

  $y_1 = e^{0x}\sin(x) = \sin(x)$ $y_2 = e^{0\alpha}\cos(1x) = \cos(x)$

- Ex: $y'' - y' - 6y = 0$, $r^2 - r - 6 = 0$, $(r+2)(r-3) = 0$, $r = -2, 3$

  $y_1 = e^{-2x}$, $y_2 = e^{3x}$ so $y = c_1 e^{-2x} + c_2 e^{3x}$

- The higher order case: $a_n y^n + \ldots + a_2 y' + a_0 y = 0$

  $a_n, \ldots a_2, a_0$ are all constants

  Char. poly.: $a_n r^n + \ldots a_1 r + a_0 = 0 = P(r)$

  factorize $P(r)$ into linear and quad terms $P(r) = a_n (r - r_1)^{m_1} \ldots (r - r_k)^{m_k} (r^2 - 2\alpha_1 r + \alpha_1^2 + \beta_3^2)^{n_1}$

  $\ldots (r^2 - 2\alpha_e r + \alpha_e^2 + \beta_e^2)^{n_e}$, each factor contrib tes some sol., the general sol. is just the

  linear comb. of them all, y

  * $(r - r_1)^{m_1} \Rightarrow e^{r_1 x}, xe^{r_1 x}, \ldots, x^{m_1 - 1} e^{r_1 x}$

  ** $(r^2 - 2\alpha r + \alpha^2 + \beta^2)^n \Rightarrow y = (c_1 + c_2 x + \ldots + c_n x^{n-1}) e^{\alpha x}\cos(\beta x) + (d_1 + d_2 x + \ldots + d_n x^{n-1}) e^{\alpha x}\sin(\beta x)$

Ex: $y''' - 4y'' - 5y' = 0$

$r^3 - 4r^2 - 5r = 0 \quad = r(r^2 - 4r - 5) = 0 \quad = r(r-5)(r+1) = 0$ so $r = 0, 5, -1$

general solution: $y = c_1 e^{0x} + c_2 e^{5x} + c_3 e^{-x} \quad = c_1 + c_2 e^{5x} + c_3 e^{-x}$

Ex: $y''' - 5y'' + 3y' + 9y = 0$

How to find a linear factor? Rational root test

$P(r) = a_n r^n + \dots a_1 r + a_0$

$a_n \dots a_1, a_0$ are integers if $r = p/a$ is a rational root in lowest term then p divides evenly into a_0 and q divides evenly into a_n.

$a_0 = 9$, $a_n = 1 = a_3$ $\quad q = 1$, $p/q \Rightarrow p = \pm 1, \pm 3, \pm 9$ you can check that $p/q = -1$ is a root

$r^3 - 5r^2 + 3r + 9 = r^3 + r^2 - 6r^2 - 6r + 9r + 9 \quad = (r+1)(r^2 - 6r + 9) \quad = (r+1)(r-3)^2$

$y = c_1 e^{-x} + c_2 e^{3x} + c_3 x e^{3x}$