- Non-Homo. ODE
- $L = a_n D^n + a_{n-1} D^{n-1} + a_0$
  $D = d/dx$
- Consider: $Ly = g(x)$ (1)
  Solution (Super. position principal) $y = y_c + y_p$, where y_c in the general solution of $Ly = 0$ (2), y_p = particular solution
- Next lectures: learn how to find y_p!

Section 4.4 - Methods of undetermined coefficients

- General idea - guess the solution y_p from the form of $g(x)$ and y_c.
- Applicable if $g(x)$ is a sum of terms of the following forms:
  - $P(x)$: polynominal, ie; $b_n x^x + b_{n-1} x^{x-1}, \dots + b_2 x + b_0$
  - $P(x)e^{\alpha x}$
  - $P(x) e^{\alpha x} \cos(\beta x)$, $P(x) e^{\alpha x} \sin(\beta x)$
  - $+ \; y_p = x^m$ (linear sum of functions that are generated by repeated diff. of $g(x)$)
    - m: depends on how $g(x)$ is related to y_c.
- Consider $ay'' + by' + cy = g(x)$
  homo. eq: $ay'' + by' + cy = 0$
  Char. eq: $ar^2 + br + c = 0$
- Magic Box:

  Case 1 - $g(x) = P(x) e^{r_0 x}$, $P(x)$ polynomial of order k

  $$y_p = x^m \underbrace{(A_k x^k + \dots + A_1 x + A_0)}_{\text{sum of derivatives of } g(x)} e^{r_0 x}$$

  $m = 0$ if r_0 is not a root of $*$

  $m = 1$ if r_0 is a simple root of $*$ ($e^{r_0 x}$ is a part of y_c)

  $m = 2$ if r_0 is a double root of $*$

  Case 2: $g(x) = \begin{cases} P(x) e^{\alpha x} \cos(\beta x) \\ P(x) e^{\alpha x} \sin(\beta x) \end{cases}$ $p(x)$ = polynomial degree

  $$y_p = x^m \left[(A_k x^k + \dots + A_2 + A_0) e^{\alpha x} \cos(\beta x) + (B_k x^k + \dots + B_2 x + B_0) e^{\alpha x} \sin(\beta x) \right]$$

  $m = 0$, if $\alpha + i\beta$ is not a root of $*$

  $m = 1$, if $\alpha + i\beta$ is a root of $*$

Ex: $y'' - 2y' + y = 2e^x$

$g(x) = 2e^x \Rightarrow P(x) = 2$ of degree 0, $r_0 = 1$

1st step: solve homo. eq.

Char. eq.: $r^2 - 2r + 1 = 0$ has a double root $r = 1$

$\Rightarrow y_p = x^m (Ae^x)$, $m = 2$

$y_p = x^2 A e^x$ (Not final answer)

Determine A:

$$y_p' = 2Axe^x + Ax^2e^x$$
$$y_p'' = 2Ae^x + 2Axe^x + Ax^2e^x$$
$$= Ax^2e^x + 4Axe^x + 2Ae^x$$
$$y_p'' - 2y_p' + y = 2e^x$$
$$Ax^2e^x + 4Axe^x + 2Ae^x - [4Axe^x + 2Ax^2e^x] + x^2Ae^x = 2e^x$$
$$2Ae^x = 2e^x \quad \text{so} \quad A = 1$$

Ex: $y'' - 2y' + 2y = 2e^x \cos x$

$P(x) = 2$ degree 0, $\alpha = 1$, $\beta = 1$

Char. eq: $r^2 - 2r + 2 = 0$

has complex roots $r = 1 \pm i$. obscene $\alpha + i\beta = 1 + i$ is a root of the char eq.

$y_p = x^m (Ae^x \cos x + Be^x \sin x)$, $m = 1$ so, $y_p = x(Ae^x \cos x + Be^x \sin x)$

Next step: determine A and B

· Ex: $y'' - 4y' + 3y = x^2 + x - 1 + \sin x$

$g(x) = \underbrace{x^2 + x - 1}_{p(x)} + \underbrace{\sin(x)}_{\sin x} = g_1(x) + g_2(x)$

use $g_1(x)$ to guess y_{p_1} and you use $g_2(x)$ to guess y_{p_2}

$y_p = y_{p_1} + y_{p_2}$, $g_1(x) = x^2 + x - 1$, $p(x) = x^2 + x - 1$ degree 2 , $r_0 = 0$

Char. eq. : $r^2 - 4r + 3 = 0$, has 2 roots , $r = 1, 3$

$y_{p_1} = x^m(Ax^2 + Bx + c)$ $m = 0$, $y_{p_1} = Ax^2 + Bx + c$

$g_2(x) = \sin x$, $p(x) = 1$, $\alpha = 0$, $\beta = 1$

$\alpha + i\beta$ is not a root

$\Rightarrow$ $y_{p_2} = x^m(D \sin x + E \cos x)$, $m = 0$

$y_{p_2} = D \sin x + E \cos x$

$y_p = Ax^2 + Bx + c + D \sin x + E \cos x$

· Section 4.6 – Variation of parameters

· Another way to find y_p

· y_p is determined by $g(x)$ and y_c in a concrete way

· consider 2nd order linear equation: $a_2(x)y'' + a_1(x)y' + a_0(x)y = g(x)$ $\Leftrightarrow$ $y'' + p(x)y' + q(x)y = f(x)$

the associated homo. eq. $y'' + py' + qy = 0$, $y_c = c_1 y_1 + c_2 y_2$

Assume that y_1 and y_2 are known then $y_p = u y_1 + v y_2$, u and v are computed as:

$W = \det\begin{pmatrix} y_1 & y_2 \\ y_1' & y_2' \end{pmatrix} \neq 0$ if linearly Independ.

$w_1 = \det\begin{pmatrix} 0 & y_2 \\ f(x) & y_2' \end{pmatrix}$, $w_2 = \det\begin{pmatrix} y_1 & 0 \\ y_1' & f(x) \end{pmatrix}$

$u = \int w_1/w$, $v = \int \frac{w_2}{w}$

How to remember : replacing a column in w by the nonhomo. part)

· Ex: $y'' + y = \sec(x) = \frac{1}{\cos x}$

Method of undetermined coefficients doesn't work

char eq: $r^2 + 1 = 0$, $r = \pm i$

$y_1 = \cos x$, $y_2 = \sin x$

$w = \begin{pmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{pmatrix} = \cos^2 x - (-\sin x^2) = 1$

$w_1 = \begin{pmatrix} 0 & \sin x \\ \sec x & \cos x \end{pmatrix} = -\sin x \sec x = \frac{-\sin x}{\cos x}$

$w_2 = \begin{pmatrix} \cos x & 0 \\ -\sin x & \sec x \end{pmatrix} = \cos x \cdot \sec x = 1$

$u = \int \frac{w_1}{w} = \int \frac{-\sin x}{\cos x} dx = \int \frac{ds}{s} = \ln(s)$ $(s = \cos x,\ ds = -\sin x\, dx)$ $= \ln|\cos x|$

$v = \int \frac{w_2}{w} = \int \frac{1}{1} = \int 1\, dx = 1$, so $y_p = u y_1 + v y_2 = (\ln|\cos x|) \cos x + x \sin x$