12.3 The Chain Rule

Composite Function

Let \( f \) and \( g \) be functions. The composite function, or composition, of \( g \) and \( f \) is the function whose values are given by \( g[f(x)] \) for all \( x \) in the domain of \( f \) such that \( f(x) \) is in the domain of \( g \).

If we define \( F(x) = (f \circ g)(x) \)
\[ = f(g(x)) \]
then,
\[ F'(x) = f'(g(x))g'(x) \]

Chain Rule

If \( y \) is a function of \( u \), say \( y = f(u) \), and if \( u \) is a function of \( x \), say \( u = g(x) \), then \( y = f(u) = f[g(x)] \), and
\[ \frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} \]

Another words, if we define \( y = f(u) \) and \( u \) is a function of \( x \), then
\[ \frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} \]

Q1.

Suppose the demand for a certain brand of lamp is given by \( D(p) = -\frac{p^2}{100} + 500 \) where \( p \) is the price in dollars. If the price, in terms of the cost \( c \), is expressed as \( p(c) = 2c - 10 \), find (a) the demand in terms of the cost, and (b) the rate of change in the demand for the lamp per unit change in price.

Q2.

A leaking oil well off the Gulf Coast is spreading a circular film of oil over the water surface. At any time \( t \) (in hours) after the beginning of the leak, the radius of the circular oil slick (in feet) is given by \( r(t) = 5t \). Find the rate of change of the area of the oil slick with respect to time.

Let \( A(r) = \pi r^2 \) represent the area of a circle of radius \( r \).

a. Find and interpret \( A[r(t)] \).

b. Find and interpret \( \frac{dA}{dt}[r(t)] \) when \( t = 100 \).